Environmental transport phenomena: Lecture III

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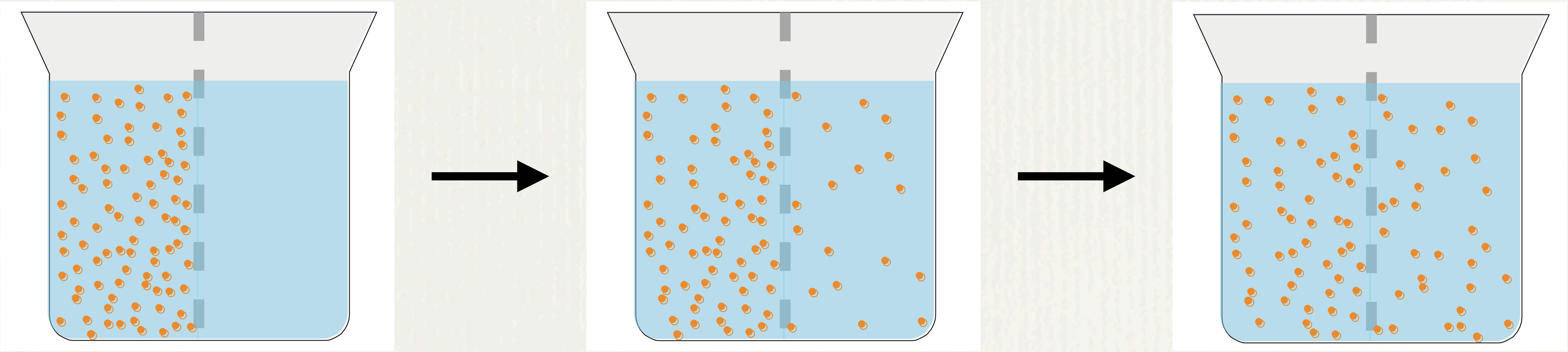
Diffusion equation

Diffusion-like equations can be found in **different contexts**:

- diffusion of substances

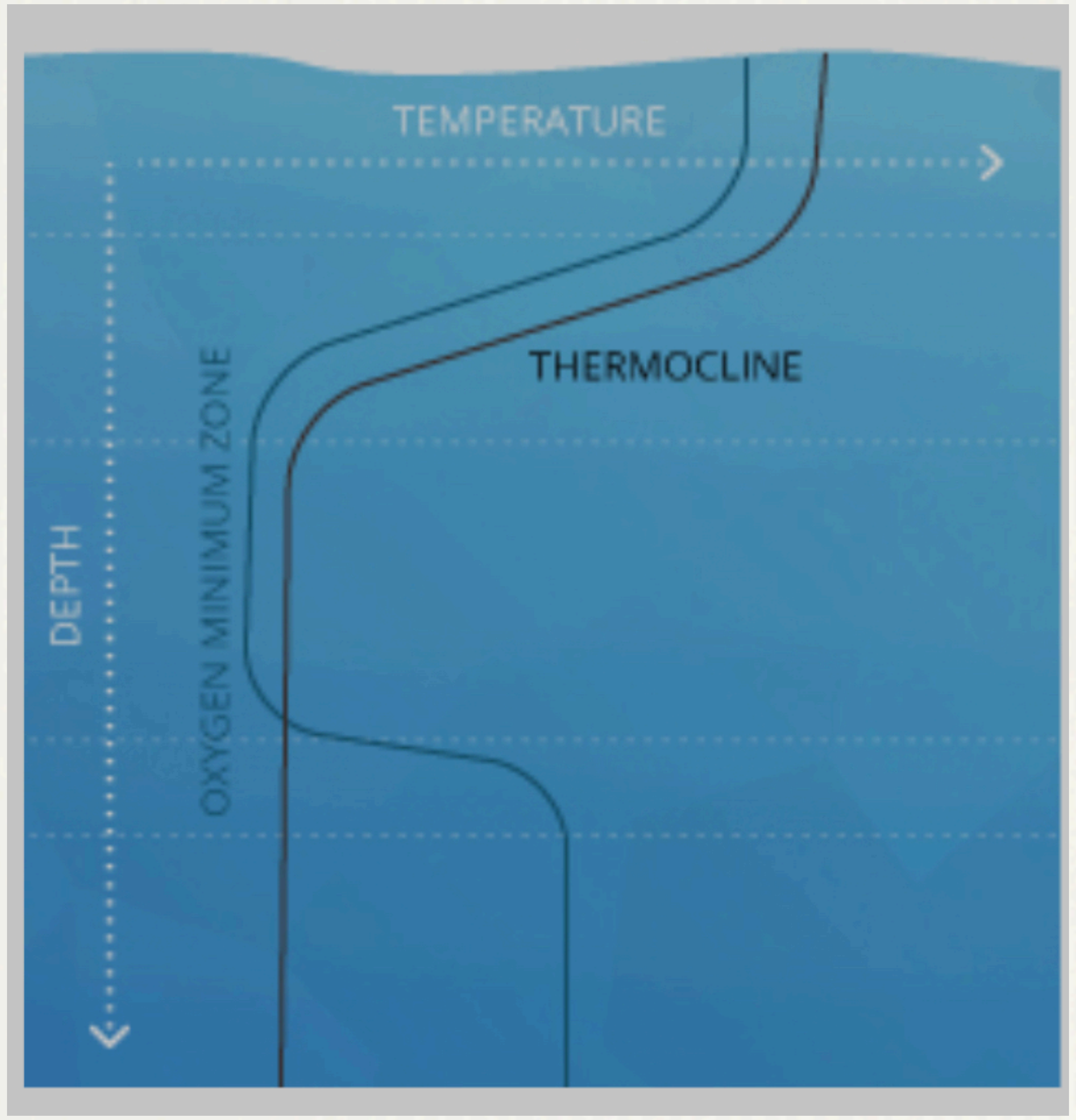
$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2}$$



Diffusion equation

Diffusion-like equations can be found in **different contexts**:

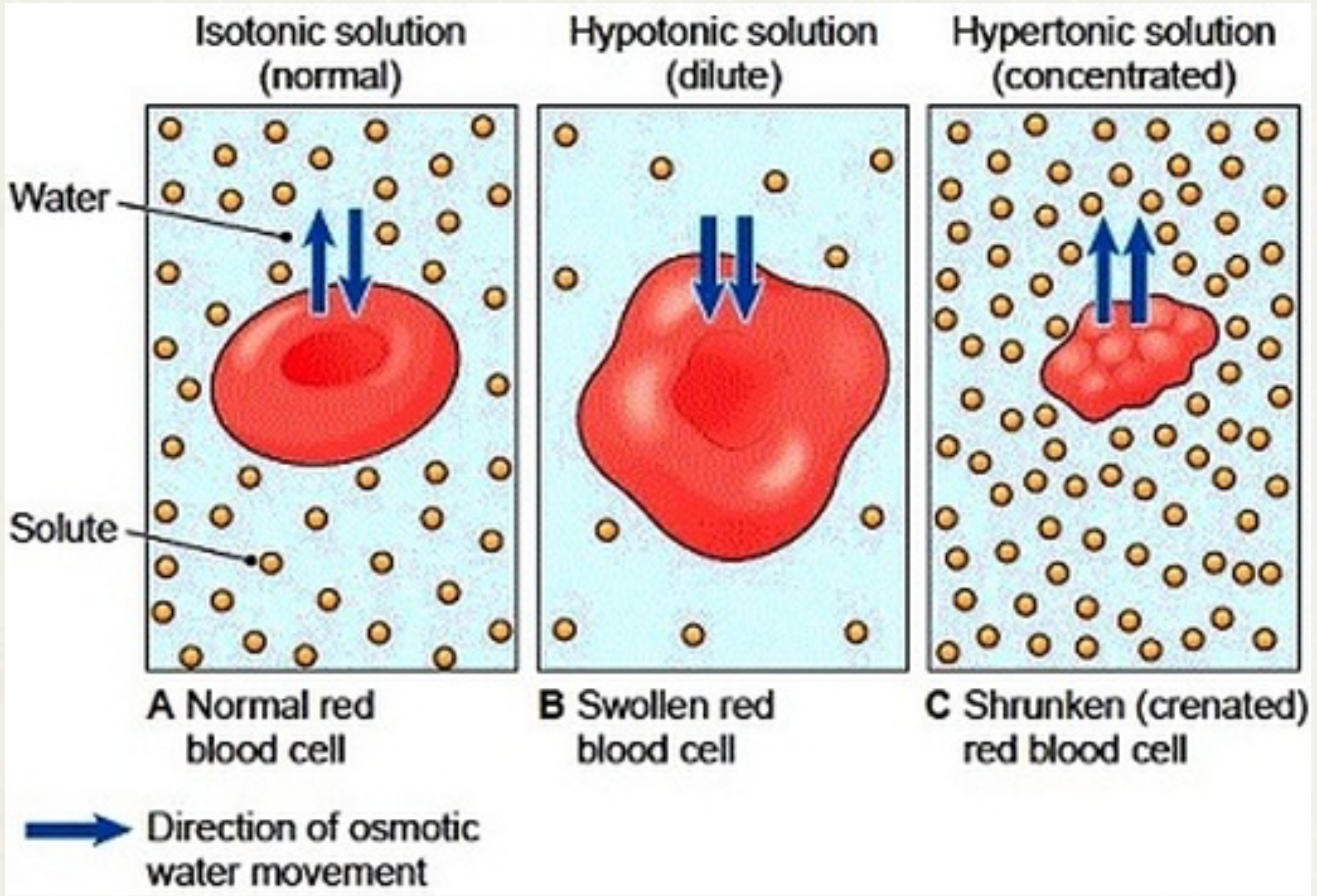
- diffusion of substances: oxygen diffusion in a water body

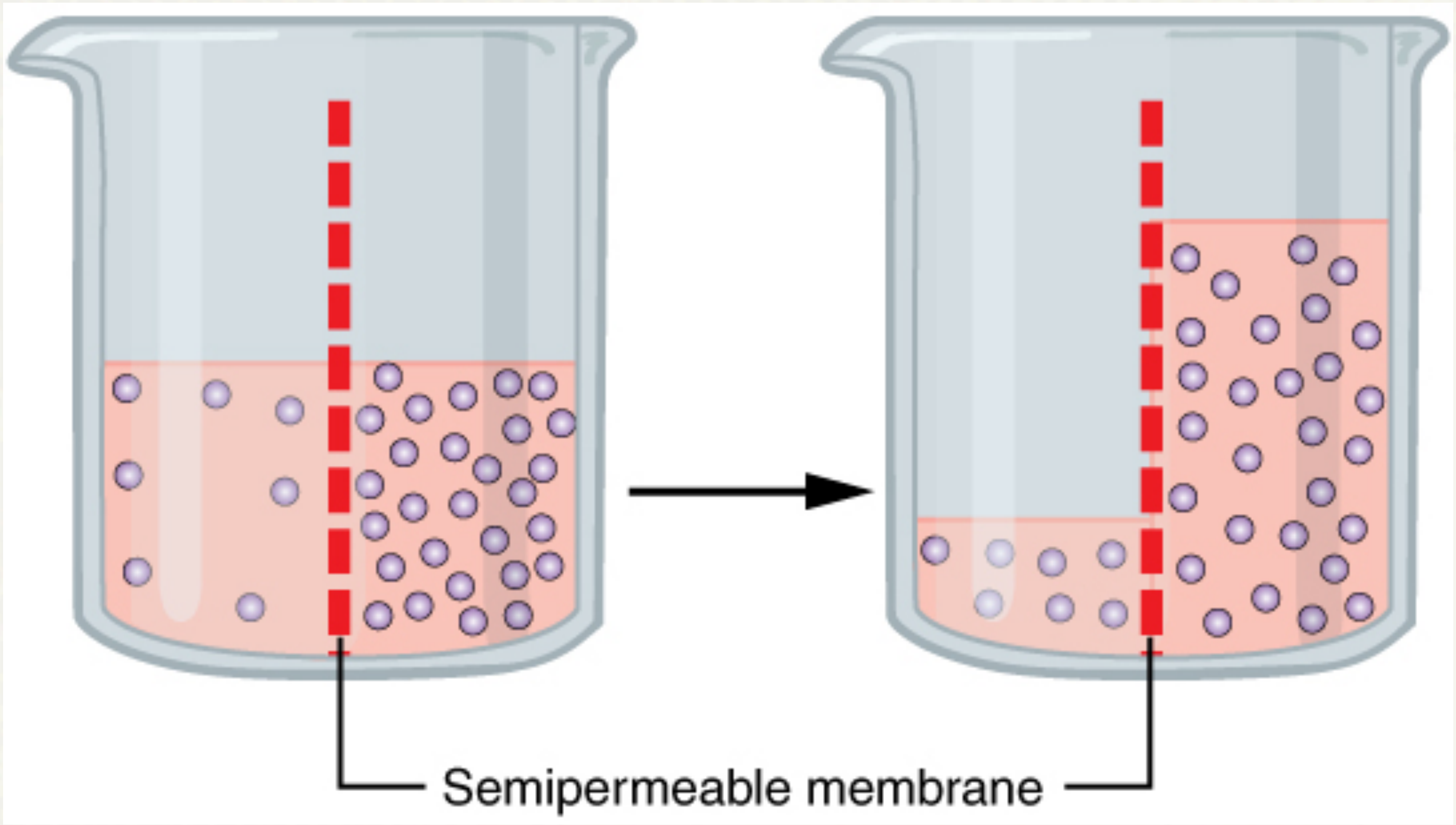


Diffusion equation

Diffusion-like equations can be found in **different contexts**:

- diffusion of substances: osmosis





Diffusion equation

Diffusion-like equations can be found in **different contexts**:

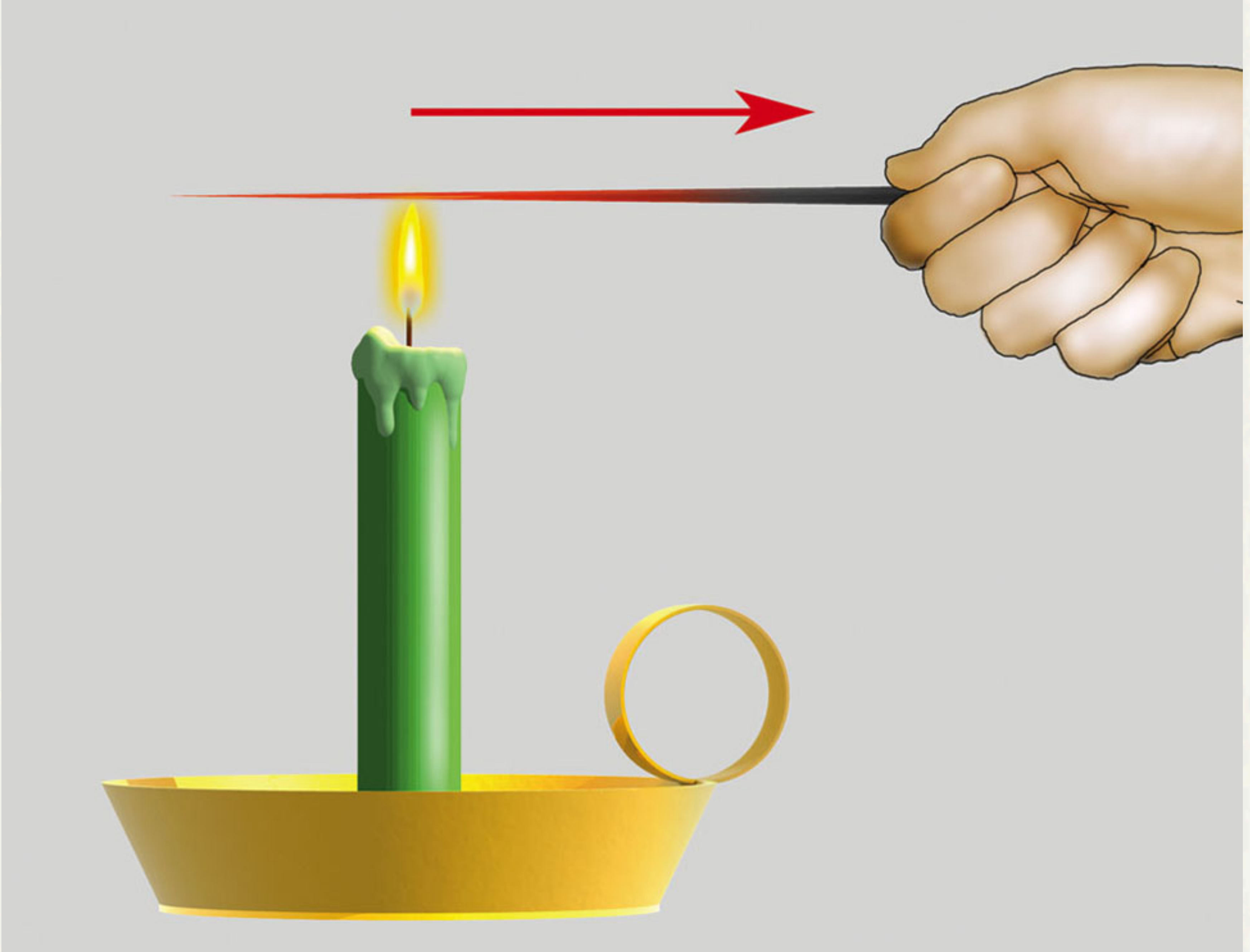
- Navier-Stokes: **momentum** diffusion

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \boxed{\mu\nabla^2\mathbf{v}} + \mathbf{f}$$

Diffusion equation

Diffusion-like equations can be found in **different contexts**:

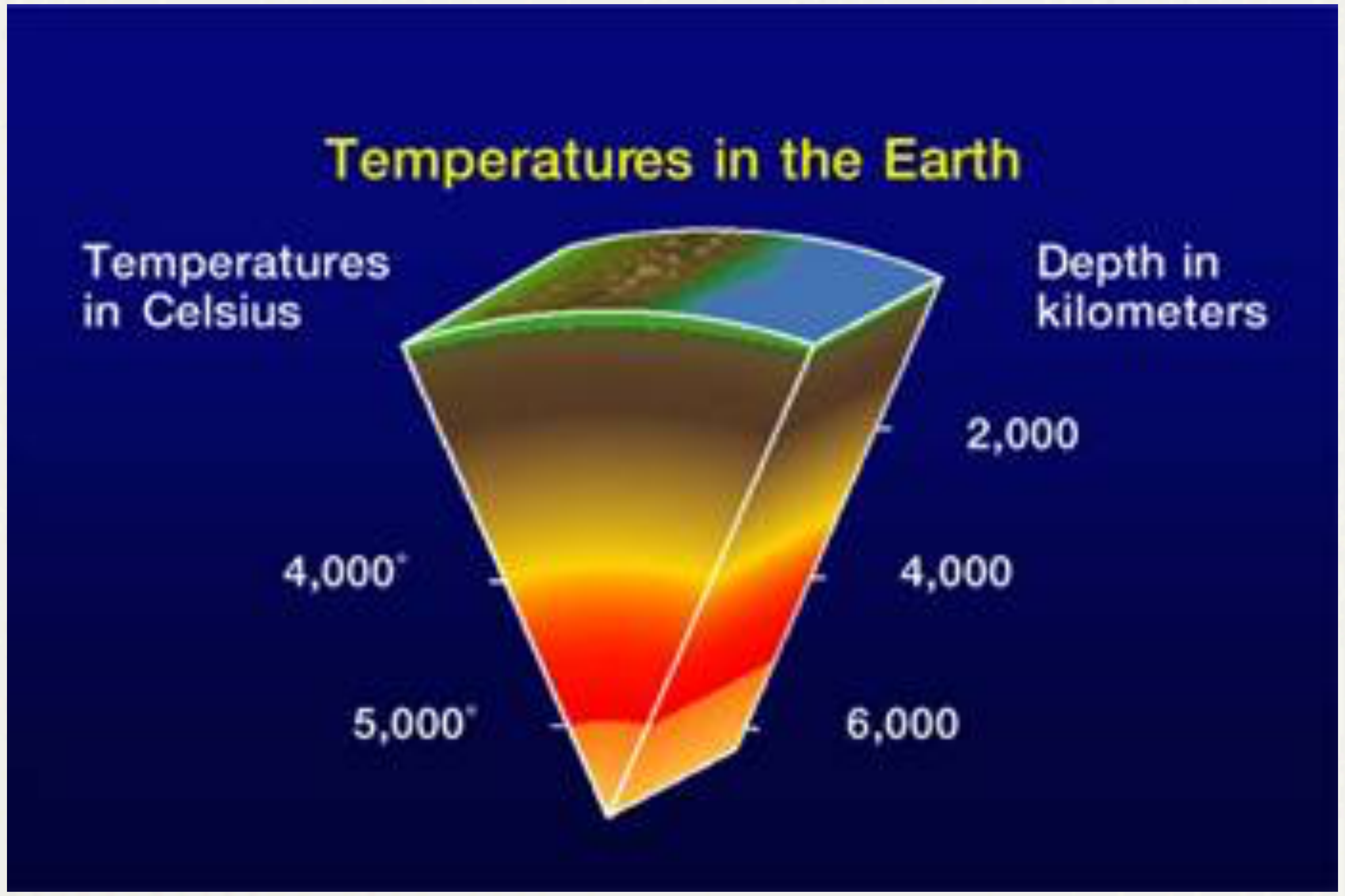
- **Heat** equation $$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$ u is the temperature and α the thermal diffusivity

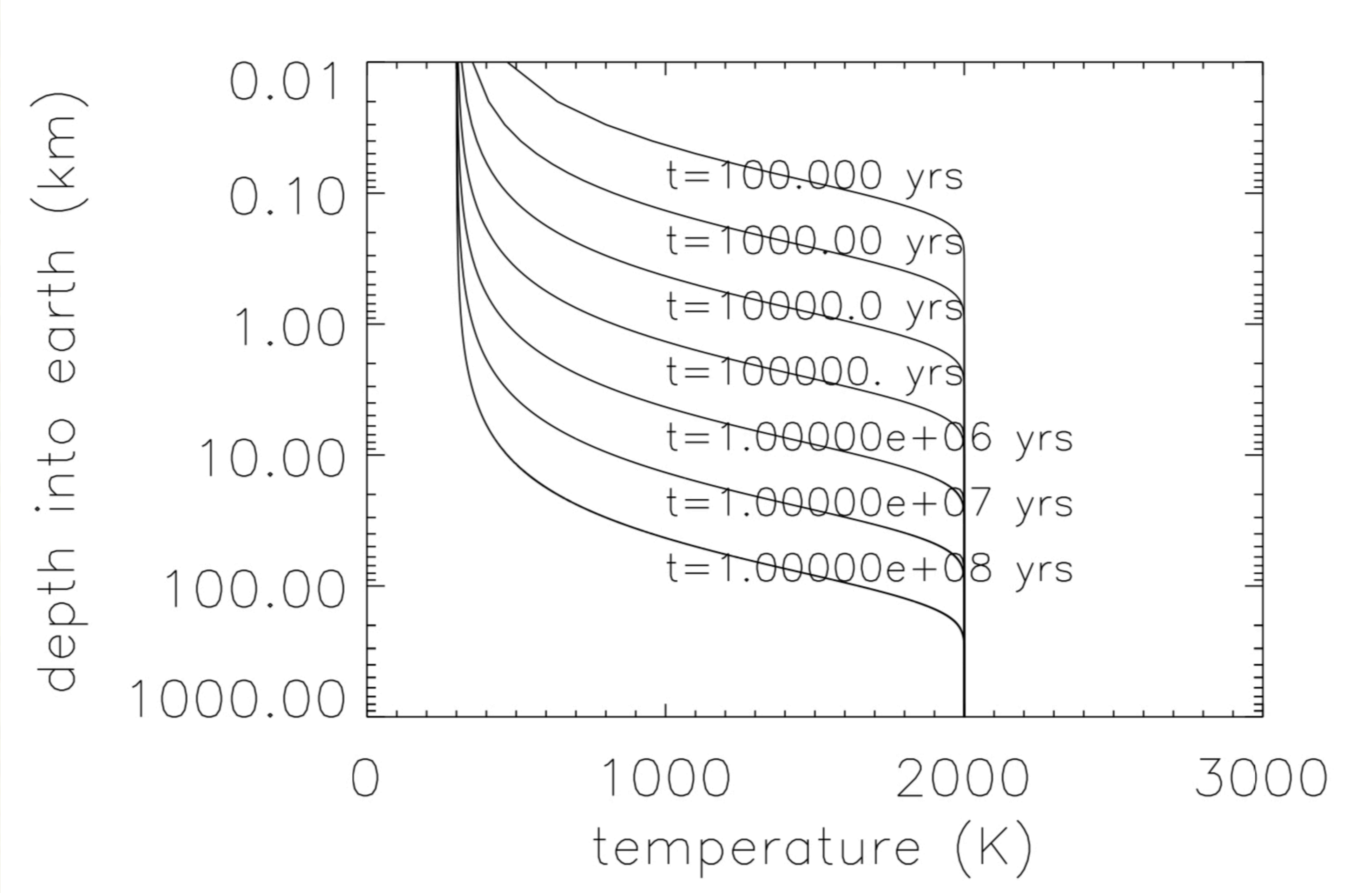


Diffusion equation

Diffusion-like equations can be found in **different contexts**:

- **Heat** equation: cooling of the earth





Diffusion equation

Diffusion-like equations can be found in **different contexts**:

- Schrödinger equation (free particle) $$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$ Diffusion in imaginary time

Diffusion equation (1D)

The diffusion equation needs to be supplemented with one **initial** condition and two **boundary** conditions. The complete problem takes the form:

- evolution equation $$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2}$$

- domain boundary concentrations $$C(x_{1,2},t) = C_{1,2} \qquad \forall t$$

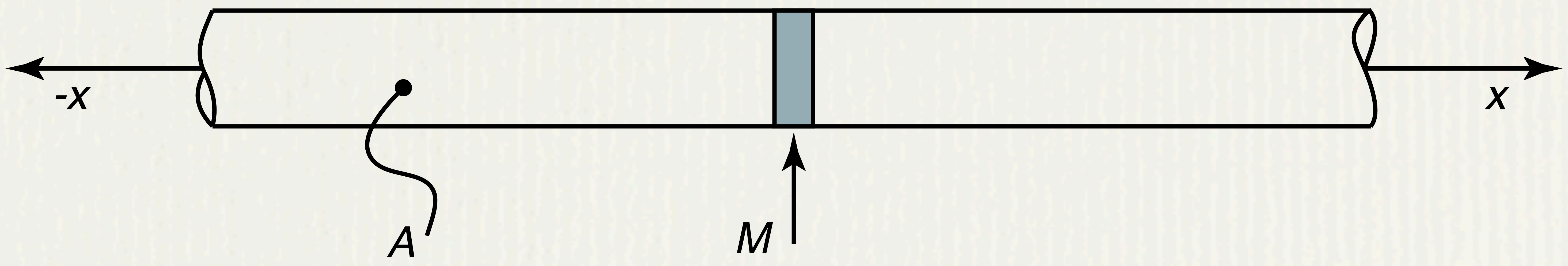
  or fluxes $$\left.\frac{\partial C(x,t)}{\partial x}\right|_{x=x_{1,2}} = C'_{1,2} \qquad \forall t$$

  domain boundaries: $x_{1,2}$

- Initial concentration distribution $$C(x,0) = f(x)$$

1D diffusion equation: point source solution

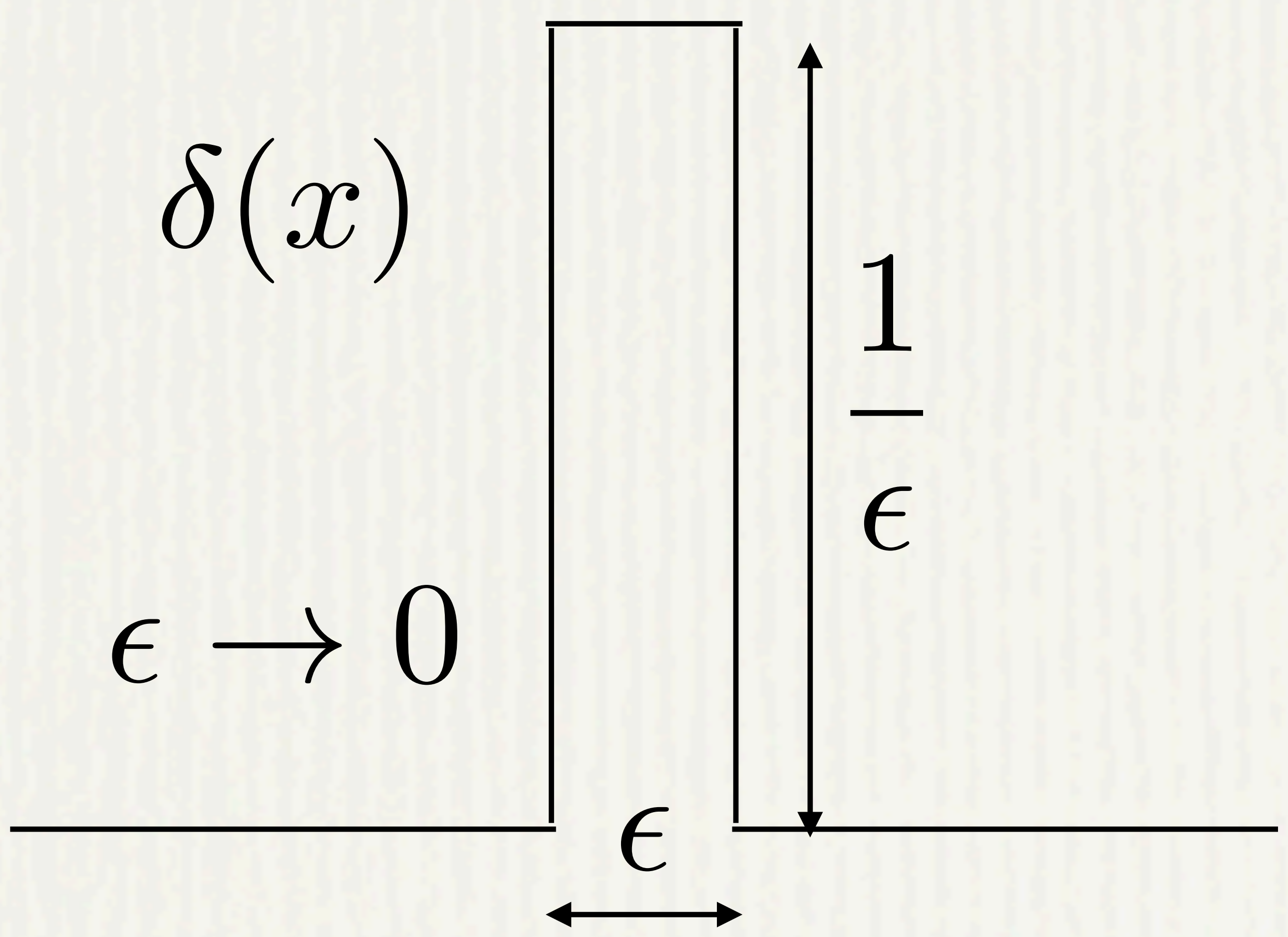
- We assume an **infinite** 1D domain and a **point** release (e.g. pipe with uniform concentration over the section A)



$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2}$$

$$C(x,t=0) = \frac{M}{A}\delta(x)$$

$$C(x=\pm\infty,t) = 0$$

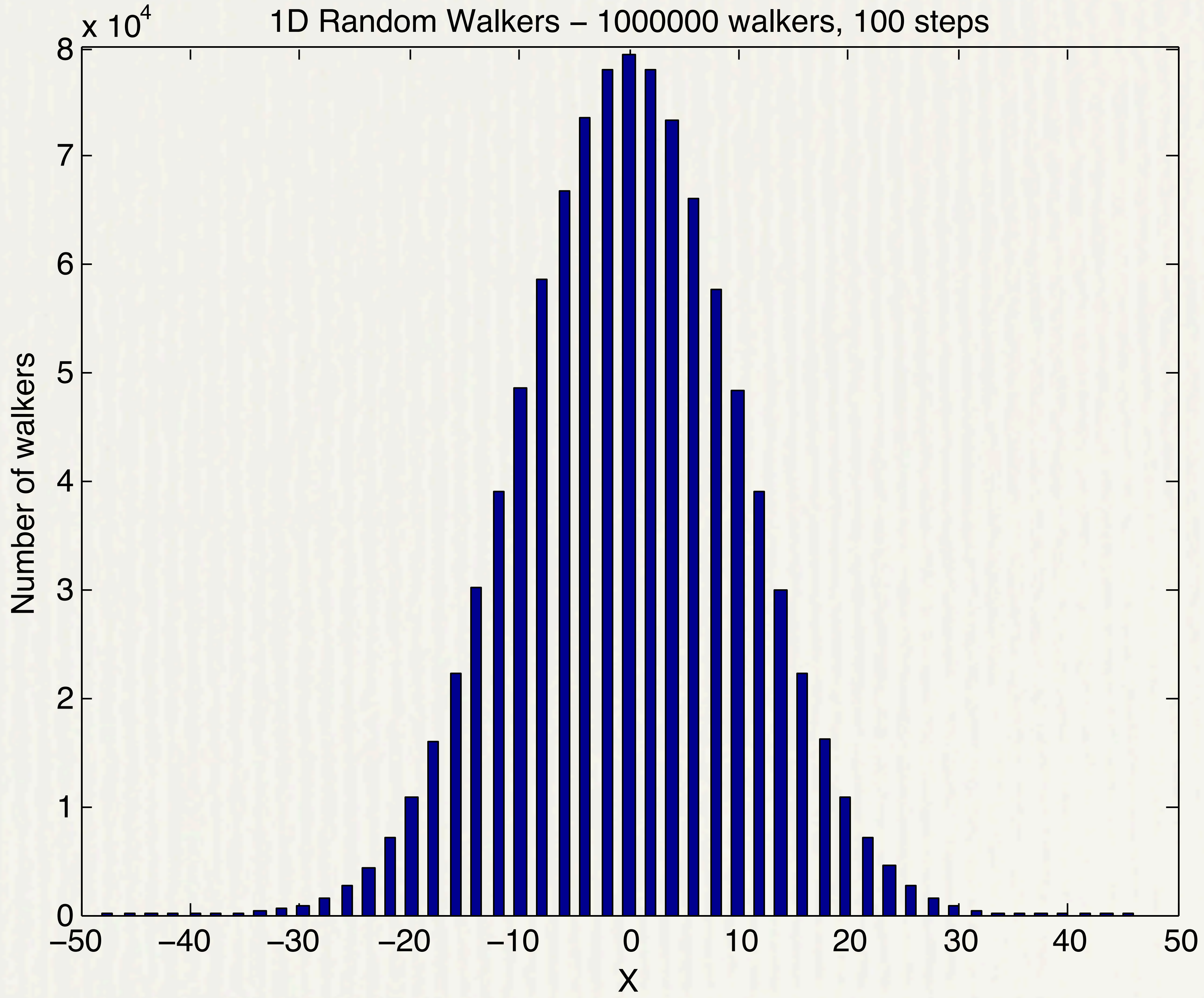


- **Dirac** distribution δ(x) satisfies (intuition: finite mass in a point):

$$\int \delta(x)dx = 1 \qquad \int \delta(x-x_0)f(x)dx = f(x_0)$$

1D diffusion equation: point source solution

$$C(x,t) = \frac{M}{A\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}} \qquad \sigma(t) = \sqrt{2Dt}$$



- Results known from normal distribution can be used: 68% of the particles found within one standard deviation, 95% within two…

- Generalisation to a point source located at position x=μ is immediate

$$C(x,t) = \frac{M}{A\sqrt{4\pi D t}} e^{-\frac{(x-\mu)^2}{4Dt}}$$