Smart Sensors for IoT
Exercise 4 (27.10.2021)

Christian Enz

Swiss Federal Institute of Technology (EPFL), Lausanne, Switzerland

Problem 1 Active Band-Pass Filter

Consider the Active Band-Pass Filter shown in Fig. 1, with an ideal amplifier, except that it is noisy (you can consider just an equivalent voltage noise source at the negative input neglecting the amplifier current noise sources at its input for the sake of simplicity) and noisy resistors:

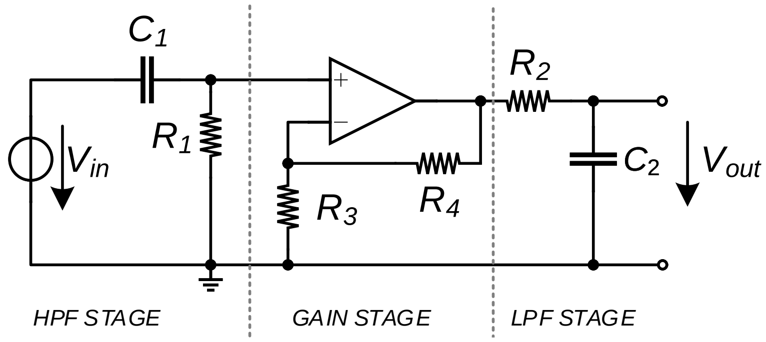

Figure 1: Active Band-Pass Filter.

Consider: $R_2 = 4R_1$, $R_3 = R_1$, $R_4 = 9R_1$ and $C_2 = 0.5C_1$ and

- derive the transfer function;
- calculate the output noise;
- calculate the input referred noise;
- calculate the output noise variance considering amplifier noise spectral density as $S_{\Delta Vn^2}$.

Problem 2 Active Band-Pass Filter variant

- What happens if the HPF stage and the LPF stage in Fig. 1 are swapped? Try to repeat the previous calculation and comment.

Problem 3 OTA-C Tow-Thomas LPF Noise Analysis

Consider the OTA-C Tow-Thomas LPF Noise Analysis shown in Fig. 2, with an ideal OTA, except that it is noisy).

Consider: $G_{m1} = G_{m2} = G_m$, $C_1 = C_2 = C$, $G_{m3} = G_m/Q$, $\omega_0 = G_m/C$, and

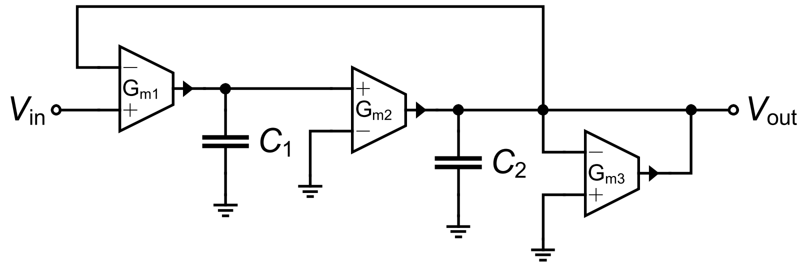

Figure 2: OTA-C Tow-Thomas LPF.

- calculate the output noise;
- calculate the input referred noise;
- calculate the output noise variance considering OTA noise spectral density as $4kT/G_{m*}$;
- finally size the components in order to synthesize a 2nd order Butter worth filter (that is, $Q = 1/\sqrt{2}$) with $\omega_0 = 2\pi \cdot 100\,\text{krad/s}$ and output noise voltage variance at room temperature ($T = 300\,\text{K}$) less than $15 \times 10^{-9}\,\text{nV}^2/\text{Hz}$.

Solutions to Exercise 4 (27.10.2021)

Problem 1 Active Band-Pass Filter

Consider the Active Band-Pass Filter shown in Fig. 1, with an ideal amplifier (except that it is noisy) and noisy resistors:

Consider: $R_2 = 4R_1$, $R_3 = R_1$, $R_4 = 9R_1$ and $C_2 = 0.5C_1$ and

- derive the transfer function;

The transfer function can be derived as the product of the transfer functions of the three stages:

$$\begin{aligned} H(s) =& H_{HPF}(s) \cdot H_{gain}(s) \cdot H_{LPF}(s) = \frac{sR_1C_1}{1+sR_1C_1} \cdot \frac{R_3+R_4}{R_3} \cdot \frac{1}{1+sR_2C_2} = \\ =& \frac{R_3+R_4}{R_3} \frac{sR_1C_1}{(1+sR_1C_1)(1+sR_2C_2)} = 10\frac{sRC}{(1+sRC)(1+s2RC)}. \end{aligned} \tag{1}$$

Let us consider the circuit shown in Fig. 3, with shorted input voltage and explicit noise sources.

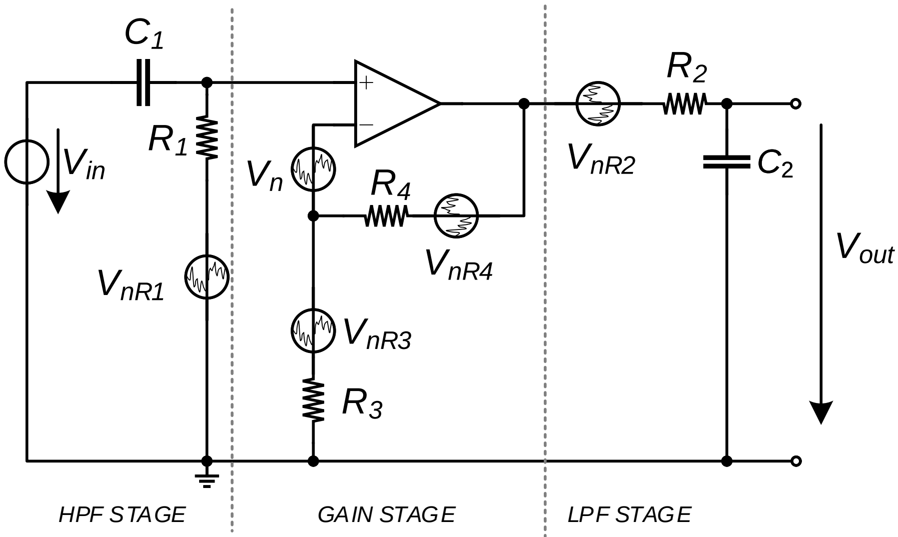

Figure 3: Active Band-Pass Filter with explicit noise sources.

Let us calculate for each sources its contribution at the output:

- V_{nR1}

Let us turn off all the sources excluding V_{nR1}. It could be noticed that in this configuration the first stage turns into a low-pass filter (the resistor is between the capacitor and the noise input and the capacitor between ground and first-stage output). Therefore in this case the transfer function can be written as:

$$H(s)_{nR1} = \frac{R_3+R_4}{R_3} \frac{1}{(1+sR_1C_1)(1+sR_2C_2)} = 10\frac{1}{(1+sRC)(1+s2RC)}; \tag{2}$$

which is a second order low-pass filter with gain 10, $Q = \sqrt{2}/3$ and $\omega_0 = 1/(\sqrt{2}RC)$, therefore the noise bandwidth times the square of the DC gain is:

$$\Delta f_{R1} \cdot |H(0)|^2_{nR1} = \frac{\omega_0 Q}{4} \cdot |H_{nR1}(0)|^2 = \frac{1}{4 \cdot 3RC} \cdot 100 = \frac{25}{3RC} \tag{3}$$

- **V_{nA}**

The noise source V_{nA} only "sees" the gain stage and the LPF stage, therefore:

$$H(s)_{nA} = \frac{R_3 + R_4}{R_3} \frac{1}{1 + sR_2C_2} = 10\frac{1}{(1 + s2RC)}. \tag{4}$$

which is a first order low-pass filter with gain 10 and $\omega_0 = 1/(2RC)$, therefore the noise bandwidth times the square of the DC gain is:

$$\Delta f_{Vn} \cdot |H(0)|^2_{nA} = \frac{\omega_0}{4} \cdot |H_{nA}(0)|^2 = \frac{1}{4 \cdot 2RC} \cdot 100 = \frac{25}{2RC} \tag{5}$$

- **V_{nR3}**

Let us call V_x the amplifier output node. Applying the KCL at the amplifier negative input node (which is at ground thanks to the virtual ground principle) leads to:

$$V_x/R_4 = V_{nR3}/R_3 \rightarrow V_x = V_{nR3} \cdot R_4/R_3. \tag{6}$$

Then, by multiplying by the LPF stage the transfer function is obtained:

$$H(s)_{nR3} = -\frac{R_4}{R_3} \frac{1}{1 + sR_2C_2} = -9\frac{1}{(1 + s2RC)}; \tag{7}$$

which is a first order low-pass filter with gain 9 and $\omega_0 = 1/(2RC)$, therefore the noise bandwidth times the square of the DC gain is:

$$\Delta f_{R3} \cdot |H(0)|^2_{nR3} = \frac{\omega_0}{4} \cdot |H_{nR3}(0)|^2 = \frac{1}{4 \cdot 2RC} \cdot 81 = \frac{81}{8RC} \tag{8}$$

- **V_{nR4} and V_{nR2}**

For both noise contributions due to R_4 and R_2, the output node of the amplifier sets respectively to V_{nR4} and V_{nR2}, therefore the two transfer functions are simply:

$$H(s)_{nR4,2} = H(s)_{nR2} = \frac{1}{1 + sR_2C_2} = \frac{1}{1 + s2RC}; \tag{9}$$

which are first order low-pass filters with gain 1 and $\omega_0 = 1/(2RC)$, therefore the noise bandwidths times the square of the DC gains are:

$$\Delta f_{R4,2} \cdot |H(0)|^2_{nR4,2} = \frac{\omega_0}{4} \cdot |H_{nR4,2}(0)|^2 = \frac{1}{4 \cdot 2RC} \cdot 1 = \frac{1}{8RC} \tag{10}$$

- Calculate the output noise.

Let us sum up all the noise contributions at the output:

$$V_{n,out}(s) = H(s)_{nR1} \cdot V_{nR1} + H(s)_A \cdot V_n + H(s)_{nR3} \cdot V_{nR3} + H(s)_{nR4} \cdot V_{nR4} + H(s)_{nR2} \cdot V_{nR2} = \\ = 10\frac{V_{nR1}}{(1+sRC)(1+s2RC)} + \frac{10V_n + 9V_{nR3} + V_{nR4} + V_{nR2}}{1+s2RC}; \tag{11}$$

- Calculate the input referred noise.

The previous expression, reported to the input, gives:

$$V_{n,in}(s) = \frac{V_{nR1}}{sCR} + \frac{1+sRC}{sCR}V_n + \frac{1}{10}\frac{(1+sRC)}{sRC}\left[9V_{nR3} + V_{nR4} + V_{nR2}\right]. \tag{12}$$

- Calculate the output noise variance.

Each noise source output variance has to be calculated as the product of the equivalent noise-bandwidth, the noise source PSD and the square of the transfer function DC gain (if any), i.e. $\overline{V_c^2} = \Delta f \cdot S_n \cdot |H(0)|^2$. We use the squared gains in this scenario as we are dealing with powers.

The output noise variance is:

$$\begin{aligned} \overline{V_c^2} =& \Delta f_{R1} \cdot |H(0)|^2_{nR1} \cdot 4kTR_1 + \\ &+ \Delta f_{R2} \cdot |H(0)|^2_{nR2} \cdot 4kTR_2 + \\ &+ \Delta f_{R3} \cdot |H(0)|^2_{nR3} \cdot 4kTR_3 + \\ &+ \Delta f_{R4} \cdot |H(0)|^2_{nR4} \cdot 4kTR_4 + \\ &+ \Delta f_{V_s} \cdot |H(0)|^2_{nA} \cdot S_{Vn} = \\ =& \frac{241kT}{3C} + \frac{25S_{Vn}}{2CR}. \end{aligned} \tag{13}$$

Problem 2 Active Band-Pass Filter variant

- What happens if the HPF stage and the LPF stage in Fig. 1 are swapped? Try to repeat the previous calculation and comment.

The band pass filter transfer function remains the same (only inverting $R_2 - C_2$ with $R_1 - C_1$, however the transfer function for the noise contributions change from all low-pass filter to high-pass filter. This means that to the output noise voltage PSDs are not frequency bounded and hence their contribution to the output noise voltage variance is *infinite*. Of course, in reality we wouldn't measure infinite noise, *why?*

Problem 3 OTA-C Tow-Thomas LPF Noise Analysis

Consider the OTA-C Tow-Thomas LPF Noise Analysis shown in Fig. 2, with an ideal OTA, except that it is noisy).

Consider: $G_{m1} = G_{m2} = G_m$, $C_1 = C_2 = C$, $G_{m3} = G_m/Q$, $\omega_0 = G_m/C$, and

- calculate the output noise;

We can decide to consider the noise sources either as a voltage source at the input of the OTA cell or as a current source at the output of cell. If we go for the former, we already have the solution by considering one input at the time a putting the other two at zero. In the following solution, however, we chose to show for the sake of completeness the case with noise current sources, as shown in Fig. 4.

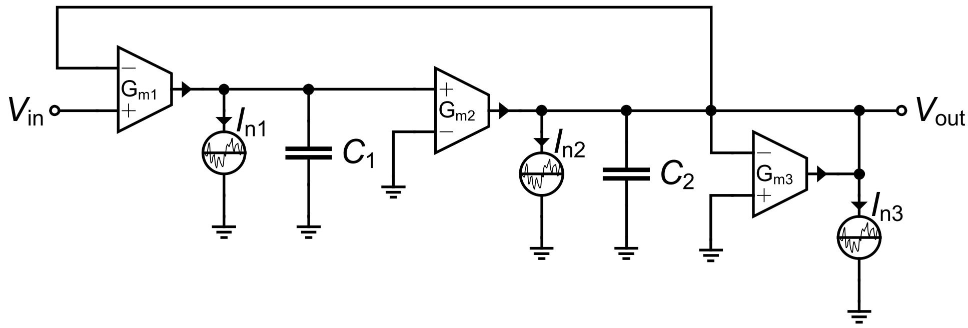

Figure 4: OTA-C Tow-Thomas LPF with current noise sources.

If we apply the superposition principle, we get the following noise voltage transimpedance transfer function at the output:

$$V_{out,n}(s) = -\frac{G_{m2}I_{n1}}{C_1C_2s^2 + C_1G_{m3}s + G_{m1}G_{m2}} - \frac{C_1I_{n2}s}{C_1C_2s^2 + C_1G_{m3}s + G_{m1}G_{m2}} - \frac{C_1I_{n3}s}{C_1C_2s^2 + C_1G_{m3}s + G_{m1}G_{m2}}. \tag{14}$$

- calculate the input referred noise;

In order to get the input referred noise, we divide (14) the the input-to-output V_{in} transfer function (found in the slides).

$$V_{in,n}(s) = -\frac{C_1I_{n2}s}{G_{m1}G_{m2}} - \frac{C_1I_{n3}s}{G_{m1}G_{m2}} - \frac{I_{n1}}{G_{m1}}. \tag{15}$$

- calculate the output noise variance considering OTA noise spectral density as $4kT/G_{m*}$;

Before calculating the output noise variance, let us replace the parameter values with the one given in the text of the exercise, that is $G_{m1} = G_{m2} = G_m$, $C_1 = C_2 = C$, $G_{m3} = G_m/Q$, $\omega_0 = G_m/C$.

The V_{in} input-to-output transfer function becomes:

$$H_0(s) = -\frac{{\omega_0}^2}{\frac{s\omega_0}{Q} + s^2 + {\omega_0}^2}. \tag{16}$$

For I_{n1} we get:

$$T_{n,1}(s) = -\frac{{\omega_0}^2}{G_m\left(\frac{s\omega_0}{Q} + s^2 + {\omega_0}^2\right)}. \tag{17}$$

For I_{n2}:

$$T_{n,2}(s) = -\frac{s\omega_0}{G_m\left(\frac{s\omega_0}{Q} + s^2 + {\omega_0}^2\right)}. \tag{18}$$

For I_{n3} we get:

$$T_{n,3}(s) = -\frac{s\omega_0}{G_m\left(\frac{s\omega_0}{Q} + s^2 + {\omega_0}^2\right)}. \tag{19}$$

We see that $T_{n,2}(s) = T_{n,3}(s)$ which is obvious since I_{n1} comes in parallel to I_{n2}. Moreover, we notice that $T_{n,1}(s)$ is a LPF transfer function, whereas $T_{n,2-3}(s)$ are BPF. Thanks to the table in the slides, we can calculate their equivalent bandwidth as:

$$\Delta f_{In,1} = \frac{Q\omega_0}{4} \qquad \Delta f_{In,2-3} = \frac{\omega_0}{4Q}, \tag{20}$$

which leads to an output noise variance for I_{n1}:

$$\overline{V^2_{c,In,1}} = \frac{4kTG_m}{G_m^2}\Delta f_{In,1} = \frac{kTQ\omega_0}{G_m} = \frac{kTQ}{C} \quad \text{with } \omega_0 = \frac{G_m}{C}. \tag{21}$$

For I_{n2}, given that the gain at resonance is $-Q/G_m$, we get:

$$\overline{V^2_{c,In,2}} = 4kTG_m\left(\frac{Q}{G_m}\right)^2 \Delta f_{In,2-3} = \frac{kTQ\omega_0}{G_m} = \frac{kTQ}{C} \quad \text{with } \omega_0 = \frac{G_m}{C}. \tag{22}$$

Similarly for I_{n3} the gain at resonance is $-Q/G_m$.We then get:

$$\overline{V^2_{c,In,3}} = 4kTG_{m3}\left(\frac{Q}{G_m}\right)^2 \Delta f_{In,2-3} = \frac{G_{m3}kTQ\omega_0}{G_m^2} = \frac{kT}{C} \quad \text{with } \omega_0 = \frac{G_m}{C} \quad \text{and } G_{m3} = \frac{Gm}{Q}. \tag{23}$$

Summing up:

$$\overline{V_c^2} = \overline{V^2_{c,In,1}} + \overline{V^2_{c,In,2}} + \overline{V^2_{c,In,3}} = \frac{kT(2Q+1)}{C}. \tag{24}$$

- finally size the components in order to synthesize a 2nd order Butter worth filter (that is, $Q = 1/\sqrt{2}$) with $\omega_0 = 2\pi \cdot 100\,\mathrm{krad/s}$ and output noise voltage variance at room temperature ($T = 300\,\mathrm{K}$) less than $15 \times 10^{-9}\,\mathrm{nV^2/Hz}$.

We can calculate the minimum C by directly replacing the values in (24):

$$\overline{V_c^2} < 15 \times 10^{-9}\,\mathrm{nV^2/Hz} \tag{25}$$

which is satisfied for $C > 1\,\mathrm{pF}$. From here we pick a value and calculate G_m in order to have the desired ω_0.