Smart Sensors for IoT
Exercise 4 (11.10.2023)

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Problem 1 Thermal Noise

Consider the circuit shown in Fig. 1, with noisy resistors R.

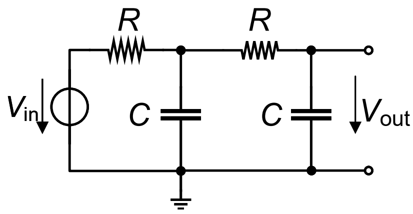

Figure 1: RC-circuit.

Calculate the variance of the thermal noise voltage across V_{out}:

- by simply solving the circuit;
- by applying the Nyquist Theorem (N.B. the derivation of the imaginary part of the output impedance and the solution of the integral can be very tedious, if you can I suggest you to use a mathematics software);
- by applying the Bode Theorem.

Problem 2 Noise of Positive Gain Stage

Consider the example of the Noise of Positive Gain Stage reported in the lecture slides (page 63) and shown in Fig. 2.

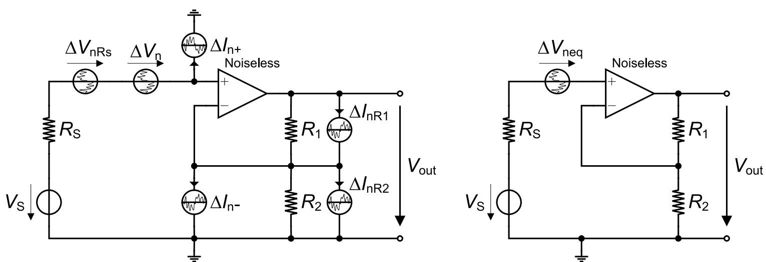

Figure 2: Noise of Positive Gain Stage

- Derive the final result of the equivalent input noise voltage PSD.

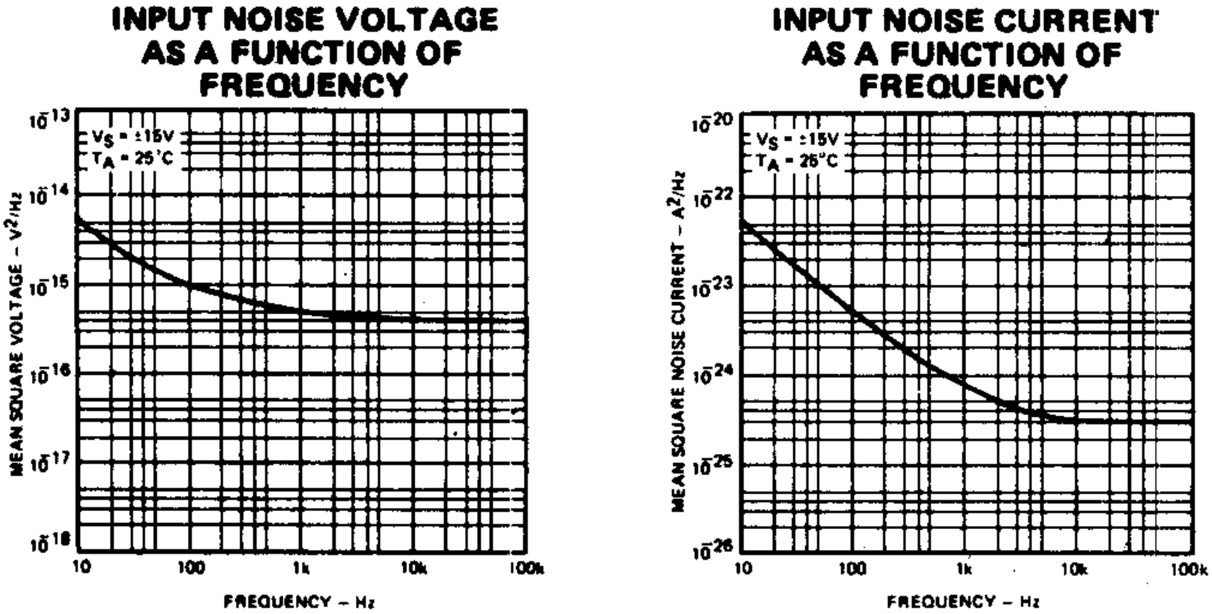

Figure 3: OpAmp noise input characteristics wrt the frequency.

- Consider $R_s = 1\,\text{k}\Omega$ and $A = 12$. The noise characteristics of the OpAmp, referred to the input is shown in Fig. 3 where the corner frequency of the input referred noise voltage is 100 Hz and its high frequency value is $3{\times}10^{-16}\,\text{V}^2/\text{Hz}$ whereas the corner frequency of the input referred noise current is 1 kHz and its high frequency is $3{\times}10^{-25}\,\text{A}^2/\text{Hz}$. Define the resistors R_1 and R_2 so that their input referred noise at the higher frequency is negligible (that is at least 10 times smaller) with respect to the source resistor R_s.
- Calculate numerically the final result of the equivalent input noise voltage PSD.

Solutions to Exercise 4 (11.10.2023)

Problem 1 Thermal Noise

Consider the circuit shown in Fig. 1, with noisy resistors R.

Calculate the variance of the thermal noise voltage across V_{out}:

- by simply solving the circuit;

In order to calculate the variance of the thermal noise voltage across V_{out}, let us short-circuit the input voltage and consider the noise sources from the two resistors as shown in Fig. 4.

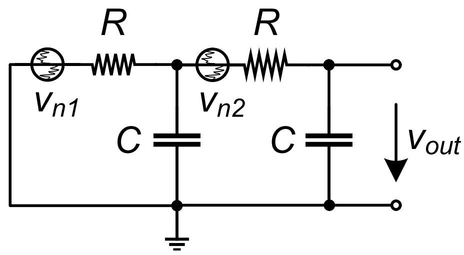

Figure 4: RC-circuit with noise sources.

Let us consider:

$$\frac{\overline{V_{n1}^2}}{\Delta f} = \frac{\overline{V_{n2}^2}}{\Delta f} = 4kTR. \tag{1}$$

In order to calculate the PSD directly, let us use the superposition effect by considering one noise source at the time and then sum up the results.

First, V_{n2} is set to 0. By applying the KCL and calling V_x the node between the resistors, the following system is obtained:

$$\begin{cases} \dfrac{V_{n,1} - V_x}{R} = V_x sC + \dfrac{V_x - V_{out}}{R}; \\ \dfrac{V_x - V_{out}}{R} = V_{out} sC; \end{cases} \tag{2}$$

from which:

$$H_1(s) = \frac{V_{out}}{V_n} = \frac{1}{1 + 3CRs + C^2R^2s^2}; \tag{3}$$

which represents a second order low pass filter transfer function with quality factor $Q = 1/3$ and cut-off frequency $f_o = 1/2\pi RC$.

According to the lecture slides, the noise bandwidth is:

$$\Delta f_1 = \frac{\omega_0 \cdot Q}{4} = \frac{1}{12RC}; \tag{4}$$

from which:

$$\overline{V_{o1}^2} = 4kTR\Delta f_1 = \frac{kT}{3C}; \tag{5}$$

Second, V_{n1} is set to 0 and the circuit can be redrawn as shown in Fig. 5 with $Z_1 = R \parallel C = \frac{R}{1+sCR}$.

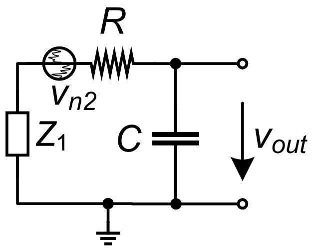

Figure 5: RC-circuit for the second resistance noise calculation

By applying the KVL to this mesh:

$$V_{n2} = V_{Z1} + V_R + V_C = i_{n2}\left(Z_1 + 1/sC + R\right); \tag{6}$$

where i_{n2} is the current flowing in the mesh.

These equations can be arranged in order to get $V_c \equiv V_{out}$ and then the transfer function:

$$H_2(s) = \frac{V_{out}}{V_n} = \frac{1 + CRs}{1 + 3CRs + C^2R^2s^2}; \tag{7}$$

which represents a second order low pass filter transfer function with a zero with quality factor $Q = 1/3$, cut-off frequency $f_o = 1/2\pi RC$ and a zero frequency $f_z = 1/2\pi RC$

According to the lecture slides, the noise bandwidth is:

$$\Delta f_2 = \frac{\omega_0 \cdot Q}{4}\left[1 + \left(\frac{\omega_0}{\omega_z}\right)^2\right] = \frac{1}{6RC}; \tag{8}$$

from which:

$$\overline{V_{o2}^2} = 4kTR\Delta f_1 = \frac{2kT}{3C}; \tag{9}$$

By summing (5) and (9),

$$\overline{V_o^2} = \overline{V_{o1}^2} + \overline{V_{o2}^2} = \frac{kT}{C}; \tag{10}$$

interestingly, this is the same result as the 1st-order RC-low-pass filter.

- by applying the Nyquist Theorem;

Let us consider the equivalent circuit shown in Fig. 6:

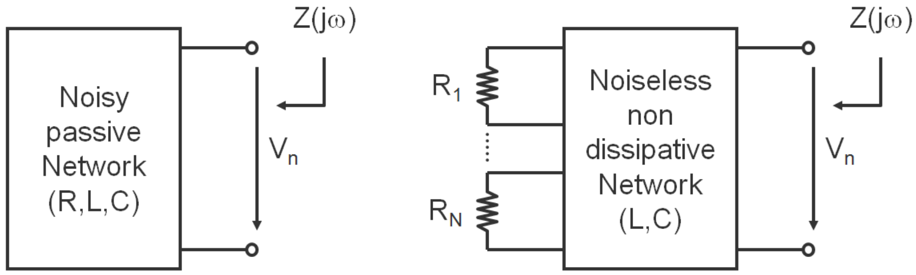

Figure 6: Nyquist equivalent.

The power spectral density (PSD) of noise voltage V_n is given by

$$S_{V_n}(f) = 4kT \cdot \Re\{Z(j2\pi f)\}; \tag{11}$$

and the variance of voltage is then given by:

$$\overline{V_n^2} = \int_0^\infty S_{V_n}(f)\,\mathrm{d}f. \tag{12}$$

Now let us calculate the output impedance Z_{out} by short-circuiting the input voltage and by applying a test current source I_x at the output of the circuit and calculating the voltage drop V_x at its terminal. The output impedance is given by:

$$Z_{out} = \frac{V_x}{I_x}. \tag{13}$$

By applying the KCL to the circuit and calling V_a the node between the resistors, and considering $Z_1 = R \parallel C = \frac{R}{1+sCR}$, the following system is obtained:

$$\begin{cases} I_x = \dfrac{V_x - V_a}{R} + V_x sC; \\ \dfrac{V_x - V_a}{R} = \dfrac{V_a}{Z_1}; \end{cases} \tag{14}$$

from which the test voltage and then the output impedance are obtained:

$$V_x = I_x \frac{R(2+sRC)}{1+3CRs+C^2R^2s^2} \rightarrow Z_{out} = \frac{V_x}{I_x} = \frac{R(2+sRC)}{1+3CRs+C^2R^2s^2}. \tag{15}$$

Let us replace s with $2\pi i f$ and calculate the real part of the output impedance:

$$S_{V_n}(f) = \Re\left\{ Z_{out}|_{s=2\pi i f} \right\} = \Re\left\{ \frac{R(2+2\pi i f RC)}{1+3CR2\pi i f - (CR_2\pi f)^2} \right\} = \frac{2R\left((2\pi \cdot R \cdot C \cdot f)^2 + 1\right)}{(2\pi \cdot R \cdot C \cdot f)^4 + 7(2\pi \cdot R \cdot C \cdot f)^2 + 1}. \tag{16}$$

Finally, by applying (12):

$$\overline{V_n^2} = 4kT \int_0^\infty \frac{2R\left((2\pi \cdot R \cdot C \cdot f)^2 + 1\right)}{(2\pi \cdot R \cdot C \cdot f)^4 + 7(2\pi \cdot R \cdot C \cdot f)^2 + 1}\,\mathrm{d}f = \frac{kT}{C} \tag{17}$$

- by applying the Bode Theorem.

The variance of noise voltage V_n can be obtained without computing the integral by using the Bode theorem stating:

$$\overline{V_n^2} = kT\left[\frac{1}{C_\infty} - \frac{1}{C_0}\right] \tag{18}$$

where

$$\frac{1}{C_\infty} = \lim_{s\to\infty} s \cdot Z_{out}(s); \qquad \frac{1}{C_0} = \lim_{s\to 0} s \cdot Z_{out}(s); \tag{19}$$

where Z_{out} has been previously calculated in (15), therefore:

$$\frac{1}{C_\infty} = \lim_{s\to\infty} s \cdot Z_{out}(s) = \lim_{s\to\infty} s \cdot \frac{R(2+sRC)}{1+3CRs+C^2R^2s^2} = \frac{1}{C}; \tag{20}$$

$$\frac{1}{C_0} = \lim_{s\to 0} s \cdot Z_{out}(s) = \lim_{s\to 0} s \cdot \frac{R(2+sRC)}{1+3CRs+C^2R^2s^2} = 0. \tag{21}$$

Finally:

$$\overline{V_n^2} = kT\left[\frac{1}{C_\infty} - \frac{1}{C_0}\right] = kT/C. \tag{22}$$

Problem 2 Noise of Positive Gain Stage

Consider the example of the Noise of Positive Gain Stage reported in the lecture slides (page 63) and shown in Fig. 2.

- Derive the final result of the equivalent input noise voltage PSD.

Let call A the gain of the circuit, defined as $A = (R_1 + R_2)/R_2$.

A way to calculate the output noise is to use the superposition principle, i.e. consider only one source at the time and set the other source zero and then sum up the results.

It could be noticed that the V_{nRs}, V_n and $V_{In+} = R_s \cdot I_{n+}$ are multiplied by the gain at the output, obtaining:

$$H_{VnRs} = A \cdot V_{nRs}; \qquad H_{Vn} = A \cdot V_n; \qquad H_{In+} = A \cdot I_{n+} \cdot R_s. \tag{23}$$

When considering the noise contributions on the right side, all the voltage sources on the left side are shorted and the current source is opened, therefore the negative input of the amplifier is at ground since no current is flowing in its branch. Thanks to the virtual ground principle, also the positive input of the amplifier is at 0V.

When considering the noise contribution coming from I_{n+}, it should be noticed that R_2 is between two ground voltages therefore no current flows through it. This means that all the current generated by I_{n+} flows through R_1 and, therefore, the output voltage is given by:

$$H_{In-} = I_{n-} \cdot R_1. \tag{24}$$

The same reasoning applies to I_{nR1} and I_{nR2} :

$$H_{R1} = I_{nR1} \cdot R_1; \qquad H_{R2} = I_{nR2} \cdot R_1. \tag{25}$$

The sum of all the transfer functions gives the output voltage noise.

If everything is reported at the input (i.e. it is dived by the gain) the equivalent noise source V_{neq} is obtained:

$$V_{neq} = \frac{1}{A}\left(H_{VnRs} + H_{Vn} + H_{In+} + H_{In-} + H_{R1} + H_{R2}\right) = V_{nRs} + V_n + I_{n+} \cdot R_s + R_{12} \cdot \left(I_{n-} + I_{nR1} + I_{nR2}\right); \tag{26}$$

where $R_{12} = \frac{R_1}{A} = \frac{R_1 \cdot R_2}{R_1 + R_2} \equiv R_1 \parallel R_2$.

Let us now calculate the input referred noise voltage PSD derived from (26) and given by:

$$S_{\Delta V^2_{neq}} = S_{\Delta V^2_{Rs}} + S_{\Delta V^2_n} + R_s^2 S_{\Delta I^2_{n+}} + R_{12}^2 \cdot \left(S_{\Delta I^2_{In-}} + S_{\Delta I^2_{nR1}} + S_{\Delta I^2_{nR2}}\right). \tag{27}$$

By replacing

$$S_{\Delta V^2_{Rs}} = 4kTR_S; \qquad S_{\Delta I^2_{nR1}} = \frac{4kT}{R_1}; \qquad S_{\Delta I^2_{nR2}} = \frac{4kT}{R_2}; \tag{28}$$

the final result is obtained:

$$S_{\Delta V^2_{neq}} = 4kT\left(\frac{{R_{12}}^2}{R_1} + \frac{R_{12}^2}{R_2} + R_S\right) + R_{12}^2 S_{\Delta I^2_{n-}} + R_S^2 S_{\Delta I^2_{n+}} + S_{\Delta V^2_n}. \tag{29}$$

- Consider $R_s = 1\,\text{k}\Omega$ and $A = 12$. The noise characteristics of the OpAmp, referred to the input is shown in Fig. 3 where the corner frequency of the input referred noise voltage is 100 Hz and its high frequency value is $3 \times 10^{-16}\,\text{V}^2/\text{Hz}$ whereas the corner frequency of the input referred noise current is 1 kHz and its high frequency is $3 \times 10^{-25}\,\text{A}^2/\text{Hz}$. Define the resistors R_1 and R_2 so that their input referred noise at the higher frequency is negligible (that is at least 10 times smaller) with respect to the source resistor R_s.

In order to correctly size the resistor, firstly we have to make sure that their ratio gives the desired gain and secondly we have to evaluate their input referred noise

In an positive gain stage, referring to Fig. 2, the gain is given by:

$$A = 1 + \frac{R_1}{R_2}. \tag{30}$$

Since $A = 12$, we can already say that $R_1 > R_2$ and according to (25) the noise transferred from R_2 to the output is larger than the one contributed by R_1 (why?). We need therefore only to make sure that R_2 noise is negligible with respect to R_S.

From (30) we can write R_1 as:

$$R_1 = (A - 1) \cdot R_2. \tag{31}$$

R_2 input referred noise needs to be lower than the R_S counterpart:

$$R_{12}^2 \cdot S_{\Delta I_{nR2}^2} < S_{\Delta V_{Rs}^2} \rightarrow \left(\frac{R_1 \cdot R_2}{R_1 + R_2}\right)^2 \frac{4kT}{R_2} < 4kTR_S \rightarrow \frac{121}{36}kTR_2 < 4kTR_S \rightarrow R_2 < 119\,\Omega \tag{32}$$

We can therefore select $R_2 = 100\,\Omega$, which leads to $R_1 = 1100\,\Omega$.

- Calculate numerically the final result of the equivalent input noise voltage PSD.

Based on the noise characteristic shown in Fig. 3, we can write the noise PSD for $\Delta I_{n\pm}$ and ΔV_n as:

$$\begin{aligned} S_{\Delta I_{n+}^2} = S_{\Delta I_{n-}^2} = S_{I_n^2 0}\left(1 + \frac{f_{k,I}}{|f|}\right) \\ S_{\Delta V_n^2} = S_{V_n^2 0}\left(1 + \frac{f_{k,V}}{|f|}\right) \end{aligned} \tag{33}$$

where $S_{I_n^2 0}$ and $S_{V_n^2 0}$ are the amplifier white-noise current and voltage components and $f_{k,I}$ and $f_{k,V}$ the current and voltage corner frequencies, respectively.

By replacing the given and calculated values in (29), considering a room temperature of $T = 300\,\mathrm{K}$,

$$S_{\Delta V_{neq}^2} = 3 \times 10^{-16}\,\mathrm{V^2/Hz}\left(1 + \frac{100\,\mathrm{Hz}}{f}\right) + 3.03 \times 10^{-19}\,\mathrm{V^2/Hz}\left(1 + \frac{1\,\mathrm{kHz}}{f}\right) + 1.809 \times 10^{-17}\,\mathrm{V^2/Hz}. \tag{34}$$

Fig. 7 shows the different input referred noise and their sum, with the given gain and R_2. It can be observed that the dominant component is given by V_n, and this is often the case as its noise is amplified to the output by the stage gain and it is generally greater than the other components noise.

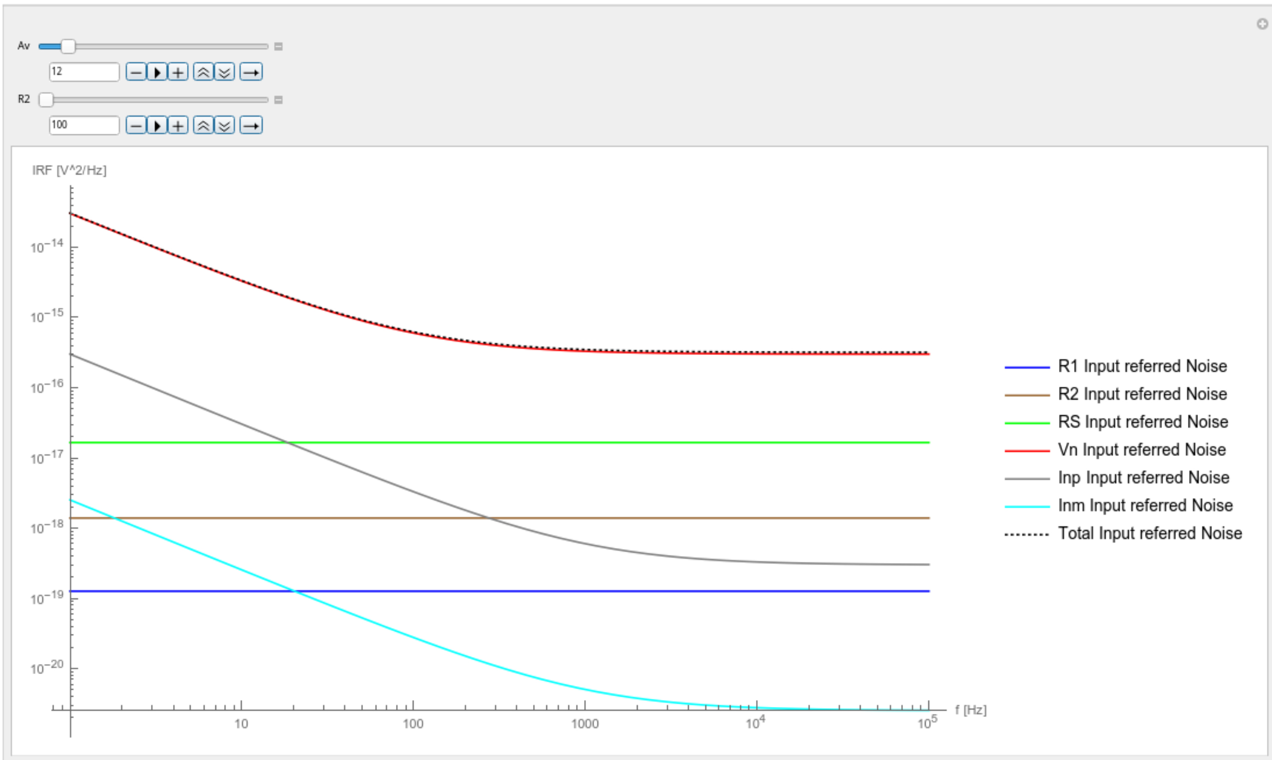

Figure 7: Equivalent noise contributions at the input.