

## Proofs of LEO approximations

Under the assumption that we are in LEO and that all changes are negligible with respect to their magnitude (ie  $\frac{\Delta A}{A} < 10^{-3} \forall A \in (r, x, v)$ ). We will prove the following approximations:

A)  $\Delta x \approx 3\pi\Delta r$

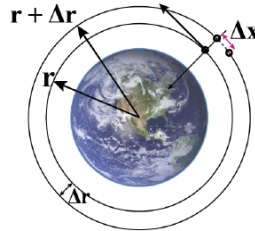
B)  $\frac{\Delta r}{r} \approx 4\frac{\Delta v}{v}$

*Hints:*

- $\sqrt{1+x} \approx 1 + \frac{x}{2} \forall x | x \ll 1$
- $\frac{1}{1+x} \approx 1 - x \forall x | x \ll 1$

**Solution.**

A) **Approach 1.** In the expression  $\Delta x \approx v\Delta T$ , one can determine the  $\Delta T$  easily assuming that the change in  $x$  produces a small change in  $r$ . We can therefore only keep the first order in  $\Delta r$ :



$$\begin{aligned}
 T(r) &= 2\pi\sqrt{\frac{r^3}{\mu}} \\
 T(r + \Delta r) &= 2\pi\sqrt{\frac{(r + \Delta r)^3}{\mu}} \approx 2\pi\sqrt{\frac{r^3 + 3r^2\Delta r}{\mu}} = 2\pi\sqrt{\frac{r^3}{\mu}}\sqrt{1 + 3\frac{\Delta r}{r}} \approx T(r)\left(1 + \frac{3}{2}\frac{\Delta r}{r}\right) \\
 \Rightarrow \Delta T &= T(r + \Delta r) - T(r) \approx \frac{3}{2}\frac{\Delta r}{r}T(r) \\
 \Rightarrow \Delta x &\approx v \cdot \frac{3}{2}\frac{\Delta r}{r}T(r)
 \end{aligned}$$

Moreover, knowing the expression of the orbital speed  $v = \sqrt{\frac{\mu}{r}}$ , we can replace in the above last statement:

$$\Delta x \approx \sqrt{\frac{\mu}{r}} \cdot \frac{3}{2}\frac{\Delta r}{r} 2\pi\sqrt{\frac{r^3}{\mu}} = 3\pi\Delta r$$

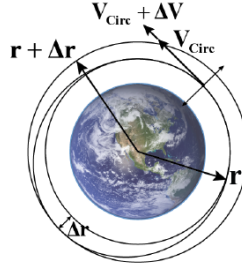
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**Approach 2.** The horizontal distance of two spacecrafts, after each orbit, is equal to the difference between the orbital paths travelled in one orbit by the two spacecrafts.

$$\begin{aligned}
 \Delta x &= C_{\text{circ}} - Tv = 2\pi(r + \Delta r) - 2\pi\sqrt{\frac{r^3}{\mu}}\sqrt{\frac{\mu}{r + \Delta r}} \\
 &= 2\pi\left(r + \Delta r - r\sqrt{\frac{r}{\Delta r + r}}\right) \approx 2\pi\left(r + \Delta r - r\sqrt{\frac{1}{1 + \frac{\Delta r}{r}}}\right) \approx 2\pi\left(r + \Delta r - r\sqrt{1 - \frac{\Delta r}{r}}\right) \\
 &\approx 2\pi\left(r + \Delta r - r\left\{1 - \frac{\Delta r}{2r}\right\}\right) = 2\pi(\Delta r + \Delta r/2) = 3\pi\Delta r
 \end{aligned}$$

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B) The semi-major axis of an orbit with a perigee at  $r$  and an apogee at  $r + \Delta r$  is given by:



$$a = \frac{r + (r + \Delta r)}{2} = r + \frac{\Delta r}{2}$$

Therefore, the expression of the velocity at the perigee is given by :

$$\begin{aligned}
 v_p &= \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}} = \sqrt{\frac{2\mu}{r} - \frac{\mu}{r + \frac{\Delta r}{2}}} = \sqrt{\frac{\mu}{r}}\sqrt{2 - \frac{1}{1 + \frac{\Delta r}{2r}}} \approx \underbrace{\sqrt{\frac{\mu}{r}}}_{=v_{\text{circ}}}\sqrt{2 - \left(1 - \frac{\Delta r}{2r}\right)} \\
 &= v_{\text{circ}}\sqrt{1 + \frac{\Delta r}{2r}} \approx v_{\text{circ}}\left(1 + \frac{\Delta r}{4r}\right)
 \end{aligned}$$

where  $v_{\text{circ}}$  is the orbital velocity for a circular orbit. The speed at the perigee of any given orbit is greater than the circular orbit ( $v_p \geq v_{\text{circ}}$ ) and therefore for a slightly eccentric orbit we have:  $v_p = v_{\text{circ}} + \Delta v$ . This yields :

$$\frac{v_p}{v_{\text{circ}}} = \frac{v_{\text{circ}} + \Delta v}{v_{\text{circ}}} \approx 1 + \frac{\Delta r}{4r} \implies \frac{\Delta r}{r} \approx 4 \frac{\Delta v}{v}$$

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