EE-585 – Space Mission Design and Operations

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Week 02 – 19 Sep 2025

Today's outline

Gravitational well

Escape and transfer velocities

Reference frames and calendars

Orbital motion and Kepler's laws

Orbits

Gravitational well

Gravitational field

An a large object of mass M exerts a force $\vec{F}_r$ on a small object of mass m at distance r

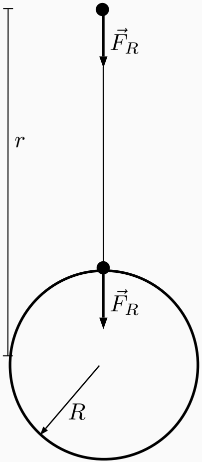
$$F_r = G\frac{Mm}{r^2} = g_r m \quad \text{Gravitational force}$$

$$g_r = G\frac{M}{r^2} = \frac{\mu}{r^2} \quad \text{Gravitational acceleration}$$

where

- $G = 6.67259 \times 10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}$ is the gravitational constant
- $\mu = GM$ is the standard gravitational parameter

The force is the same as if the whole mass was located at the center of the sphere.



On the surface of the Earth (at distance $R_\oplus$):

$$F_{R_\oplus} = G\frac{Mm}{R_\oplus^2} = g_0 m$$

$$g_0 = G\frac{M_\oplus}{R_\oplus^2} = \frac{\mu_\oplus}{R_\oplus^2}$$

$$g_0 = 9.81\ \text{m/s}^2 = 32.2\ \text{ft/s}^2$$

$$R_\oplus = 6.378 \times 10^6\ \text{m}$$

$$M_\oplus = 5.9742 \times 10^{24}\ \text{m}$$

Gravitational acceleration profile inside and outside Earth

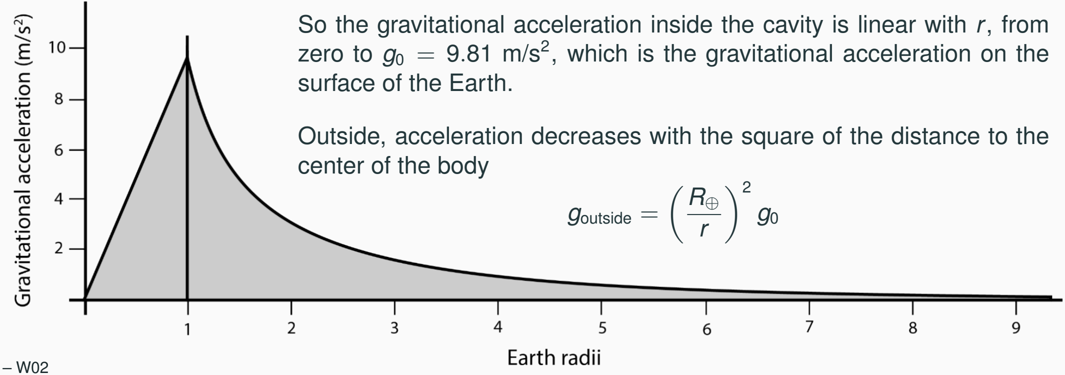
Consider a cavity at a certain distance r from the center of Earth, we determine the gravitational acceleration at this point. The shell, which is outside between r and $R_\oplus$ has a no influence, gravitationally, on any object located in this cavity.

$$g_{\text{inside}} = \left(\frac{r}{R_\oplus}\right) g_0$$

So the gravitational acceleration inside the cavity is linear with r, from zero to $g_0 = 9.81$ m/s^2, which is the gravitational acceleration on the surface of the Earth.

Outside, acceleration decreases with the square of the distance to the center of the body

$$g_{\text{outside}} = \left(\frac{R_\oplus}{r}\right)^2 g_0$$

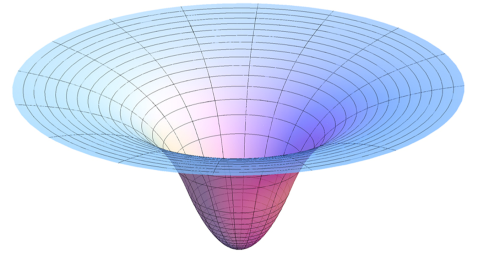


Gravitational figures for bodies in the solar system

Body	Gravity (Earth = 1)	Time to fall 4.9 m. (s)
Sun	28	0.2
Mercury	0.26	2
Venus	0.9	1.1
Earth	1	1
Moon	0.16	2.5
Mars	0.38	1.6
Phobos*	0.001	30
Jupiter	2.65	0.6
Ganymede*	0.2	2
Saturn	1.14	0.9
Titan*	0.2	2
Uranus	0.96	1
Neptune	1	1

* Approximate figures

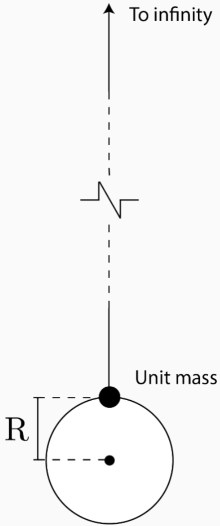
The concept of gravitational well



Credits: Wikipedia, AllenMcC

- A gravitational well is a conceptual model of the gravitational field surrounding a body.
- Entering space from the surface means climbing out of the gravitational well.
- The deeper a gravitational well is, the more energy it takes to escape from it.

Work and gravitational well from the surface



Work performed to bring a unit mass to infinity from the surface

$$W_{R_\oplus} = \int \vec{F} \cdot \mathrm{d}\vec{r} = \int_{R_\oplus}^{\infty} \frac{\mu}{r^2} \mathrm{d}r = \frac{\mu}{R_\oplus} = \frac{\mu}{R_\oplus^2} R_\oplus = g_0 R_\oplus$$

$\rightarrow$ The work necessary to lift a unit mass from the surface of Earth to infinity is the constant gravitational acceleration g_0 times the Earth's radius.

$$\boxed{W_{R_\oplus} = g_0 R_\oplus}$$

The depth of the Earth's gravitational well is equal to the radius of the Earth $R_\oplus$

Work and gravitational well from distance r

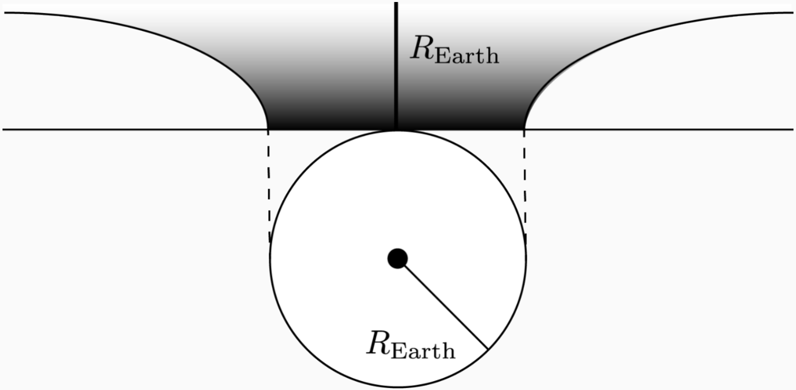
Starting at a distance r from the Earth's center, the work performed is

$$W(r) = g(r) \cdot r = \frac{g(r)}{g_0} \frac{r}{R_\oplus} W_{R_\oplus} = \frac{R_\oplus}{r} W_{R_\oplus}$$

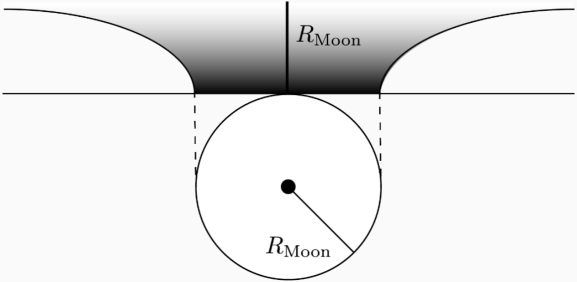
The work necessary to lift a unit mass from the distance r, larger than the radius of the Earth ($r > R_\oplus$), to infinity is equal to the work from the surface of the Earth $W_{R_\oplus}$ multiplied by the factor $R_\oplus / r$.

$\rightarrow$ The profile of the gravitational well is in $1/r$

Earth's gravitational well: depth $= R_\oplus$



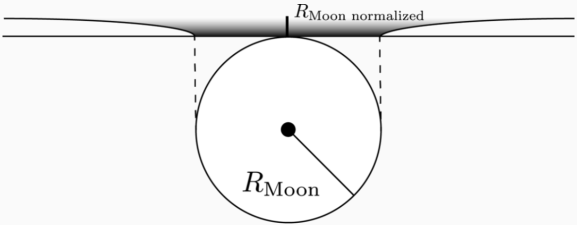
Moon's gravitational well: depth $= R_{\text{Moon}}$



The work needed to bring a unit mass from the surface of the Moon to infinity is equal to the work done to take that unit mass from the surface of the Moon to the radius of the Moon away from the Moon's surface, with a constant force equal to the gravitational force on the surface.

$$W_{\text{Moon}} = g_{0,\text{Moon}} R_{\text{Moon}}$$

Normalised gravitational well

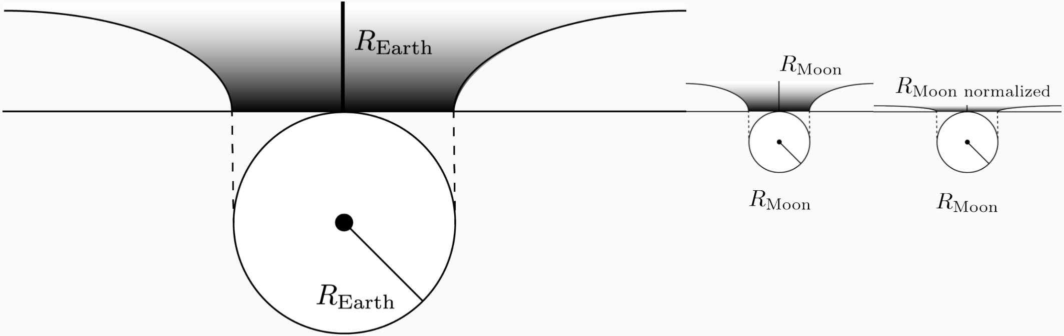


Considering that the gravitational acceleration on the Moon is only 1/6 of the value on Earth, the normalised depth of the gravitational well of the Moon is the radius of the Moon divided by 6.

The depth of gravitational well of any spherical object, is always normalised to the gravitational acceleration of the Earth for comparison purposes. It is the radius of that object, multiplied by the ratio between the gravitational acceleration on the surface of that object and the one on the surface of the Earth:

$$W_{\text{Moon}} = R_{\text{Moon}} g_{\text{Moon}} \frac{g_0}{g_0} = \underbrace{R_{\text{Moon}} \frac{g_{\text{Moon}}}{g_0}}_{\text{normalised depth}} g_0$$

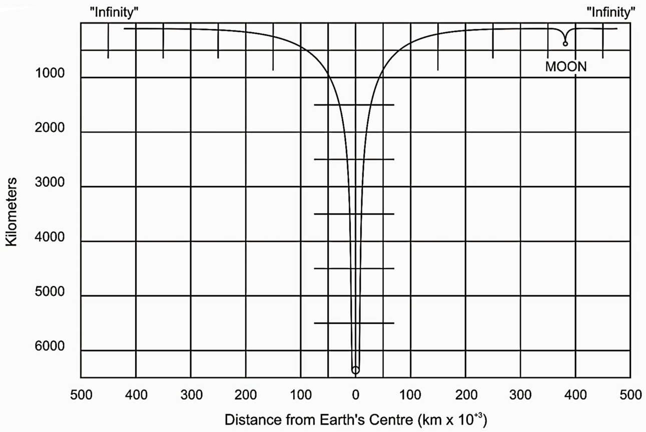
Comparison of gravitational wells



From left to right: at the same scale, gravitational well of the Earth, non-normalised gravitational well of the Moon (equal to the radius of the Moon) and normalised gravitational well of the Moon.
For a very small object like an asteroid or the nucleus of a comet, the normalised depth of the gravitational well could be equal to less than one meter or even a few centimeters.

- Moon's gravitational well depth normalised to g_0 is equal to 287 km.
- It is small in comparison with the $R_\oplus = 6378$ km and $R_{\text{Moon}} = 1738$ km.

Profile of Earth’s and Moon’s gravitational wells



Normalised profile of gravitational well of the Earth with a profile of $1/r$ and of the Moon

Credits: Adapted from “Ascent to Orbit”, Arthur C. Clarke

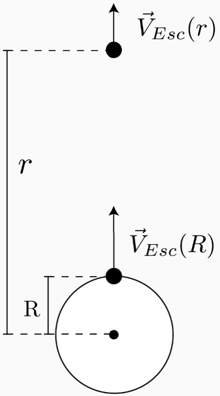
Escape and transfer velocities

Concept of escape velocity

- Escape velocity is the velocity at which a spacecraft has to leave the surface of the Earth in order to reach infinity with a zero velocity (in the absence of perturbations).
- If velocity at infinity is not zero, you have done more than what is needed to just escape the gravitational influence of the Earth.
- The work needed to bring a unit mass (the spacecraft) from the surface of the Earth to infinity is equal to the initial kinetic energy.
- How to escape from the Earth's gravitational influence?
  - A slow method that moves a unit mass in phases is inefficient.
  - Rapid, with a single impulse is the escape velocity concept, V_{Esc}.
  - Using the conservation of total energy at the Earth's surface and infinity:

$$E_{\text{tot}} = \text{cst} = E_{\text{kin}} + E_{\text{pot}} = \frac{1}{2}V^2 - \frac{\mu}{r} = \underbrace{\frac{1}{2}V_{\text{Esc}}^2 - \frac{\mu}{R_{\oplus}}}_{\text{at Earth's surface}} = \underbrace{0 + 0}_{\text{at } \infty} \implies \boxed{V_{\text{Esc}} = \sqrt{\frac{2\mu}{R_{\oplus}}}}$$

Escape velocity from distance $r > R$

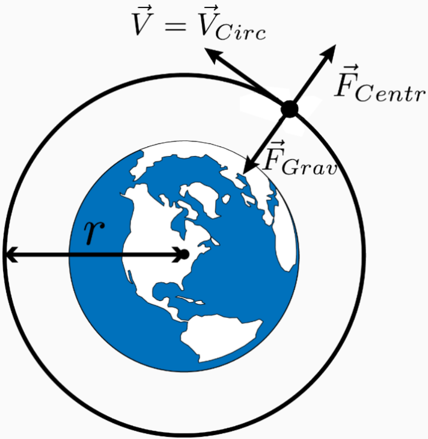


- Work performed to bring a unit mass from the Earth's surface to infinity $W = g_0 R$
- Generally, for a distance r from the centre of the Earth

$$\frac{1}{2} V_{\text{Esc}}^2(r) = g_r r$$

$$\implies V_{\text{Esc}}(r) = \sqrt{2 g_r r} = \sqrt{\frac{2\mu}{r}}$$

Circular velocity



A satellite on a circular low Earth Orbit has a velocity of the order of 7.7 to 7.8 km/s, that is going around the Earth in about 90 minutes.

The centrifugal force resulting from the curved orbital trajectory is equal in magnitude to the gravitational force on the orbiting object-

$$F_{\text{Centr}} = F_{\text{grav}}$$

$$\implies \frac{V_{\text{Circ}}^2}{r} = \frac{\mu}{r^2}$$

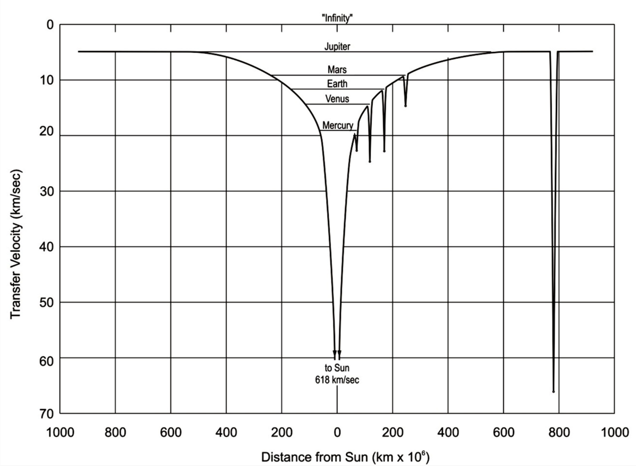
$$\implies \boxed{V_{\text{Circ}} = \sqrt{\frac{\mu}{r}}}$$

Gravitational figures for bodies in the solar system

Body	Gravity Earth = 1	Time to fall 4.9 m. (s)	Escape velocity $\sqrt{\frac{2\mu}{R}}$, km/s	Circ. velocity at surface $\sqrt{\frac{\mu}{R}}$, km/s
Sun	28	0.2	618	437
Mercury	0.26	2	3.5	2.5
Venus	0.9	1.1	10.4	7.3
Earth	1	1	11.2	7.9
Moon	0.16	2.5	2.3	1.6
Mars	0.38	1.6	5	3.6
Phobos*	0.001	30	0.01	0.01
Jupiter	2.65	0.6	60	42.5
Ganymede*	0.2	2	3	2
Saturn	1.14	0.9	36	25
Titan*	0.2	2	3	2
Uranus	0.96	1	22	15.5
Neptune	1	1	23	16

* Approximate figures

Gravitational well in term of transfer velocity



The transfer velocity, for a given planet, is the velocity that has to be added to the planet's circular velocity for a transfer to infinity from this location in the Sun's gravitational well, i.e. as if to leave the solar system.

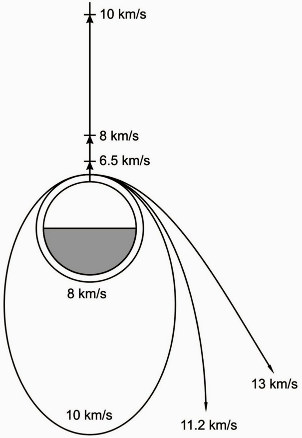
To determine the "two steps" escape velocity out of the solar system from the surface of a planet, the escape velocity from the planet itself should be added and is illustrated by the depth of the "icicles" around each planet. Typically, for the Earth the icicle has an amplitude of $V_{\text{Esc},\oplus} = 11.2$ km/s.

Credits: Adapted from "Ascent to Orbit", Arthur C. Clarke

Example of escape velocity from a planet

- Escape velocity out of the solar system from Mercury's orbit: **68 km/s.**
- Average orbital velocity of Mercury: **48 km/s.**
- Transfer velocity out of the solar system from Mercury's orbit: **68 - 48 = 20 km/s.**
- Escape velocity from the surface of Mercury: **3.5 km/s.**
- "Two steps" escape velocity out of the solar system from the surface of Mercury: **20 + 3.5 = 23.5 km/s.**

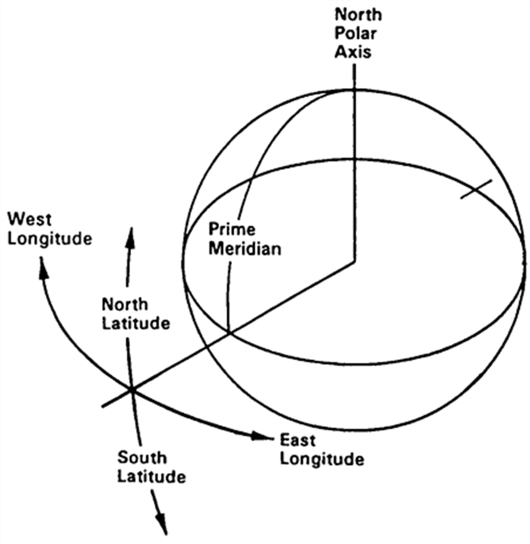
Escape velocity versus direction of escape



- The escape velocity is independent of the direction of the initial impulse (as long as escape really takes place).
- If the initial impulse is horizontal (from a point just outside of the Earth's atmosphere, or a little above 100 km altitude) :
  - If $V = V_{\text{Esc}} = 11.2$ km/s the trajectory will be parabolic
  - If $V > V_{\text{Esc}}$, the trajectory will be hyperbolic
  - If $V < V_{\text{Esc}}$, it will be an elliptical orbit (decaying because of low perigee)
  - At ~ 8 km/s the trajectory will be initially a circular orbit (but rapidly decaying)

Reference frames and calendars

Geographic coordinate system

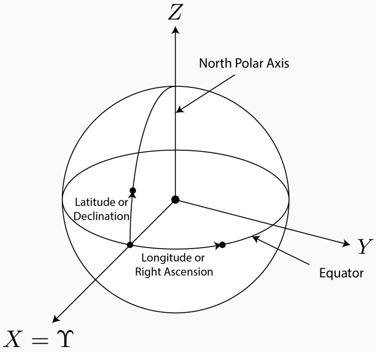


The geographic coordinate system (longitude, latitude) is used to specify a location on the surface of the Earth.

This is an Earth-centred - Earth-fixed (ECEF) frame

→ coordinates of a fixed object on Earth's surface does not depend on time.

Geocentric-inertial coordinate system (celestial coordinate system)



An inertial frame is an orthogonal frame of reference XYZ, with respect to which the laws of motion are valid.

The centre of the geocentric inertial coordinate system is the centre of the Earth.

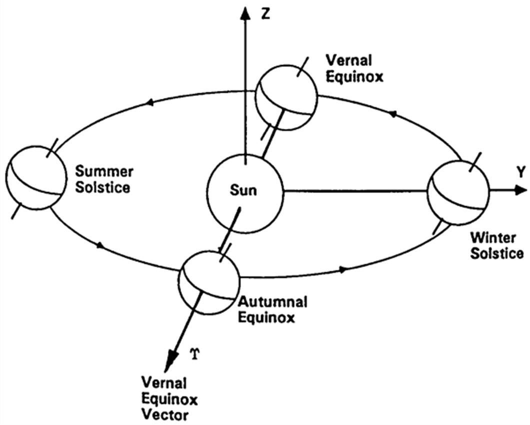
The plane of reference is the plane of the Equator, where the direction of X is the direction of the vernal equinox.

The vernal equinox ♈ is a point on the Equator that the Sun crosses when it goes from the southern celestial hemisphere to the northern celestial hemisphere around the 21st of March.

This point is very slowly migrating to the west (precession of equinoxes, about 0.014 °/yr $\implies T_{\text{precession}} \approx 26000$ yr), so, when using the geocentric-inertial coordinate system, the year shall be specified. Currently the reference is the year 2000 (epoch J2000.0).

This is an Earth-centered inertial (ECI) frame. Every 24 hours, the ECEF and ECI are aligned.

Heliocentric-inertial coordinate system



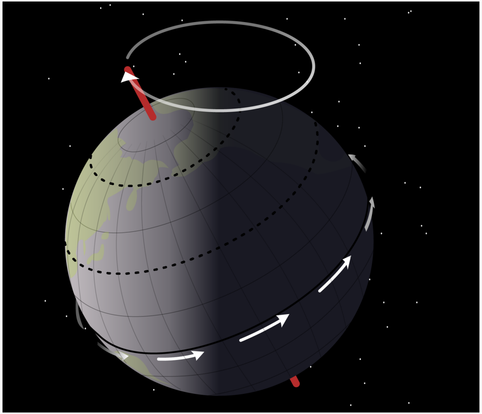
The heliocentric-inertial coordinate system has the same X-direction as the geocentric-inertial coordinate system, that is the vernal equinox ♈.

The centre of this coordinate system is in the centre of the Sun.

The plane of reference is the plane of the ecliptic, or plane of the Earth's orbit around the Sun.

You can define the axis with respect to distant astronomical objects (i.e. objects that do not have a significant motion on the sky).

Precession of the equinoxes



Credits: NASA GSFC, R. Simmon

Earth's rotational axis has a tilt of 23.5° vs. a perpendicular to the ecliptic plane.

Axial precession is the displacement of the rotational axis of an astronomical body.

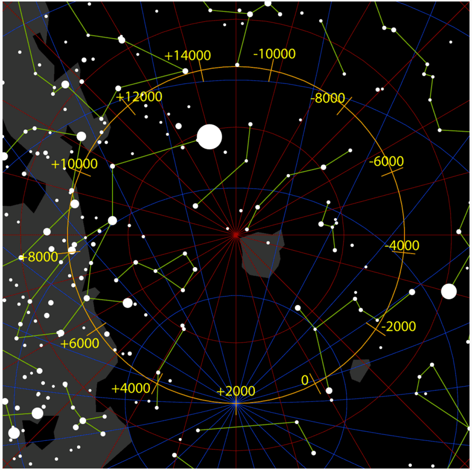
Earth goes through one such complete precessional cycle in about 26 000 years.

The precession of the equinoxes induces a difference between the solar year and the sidereal year.

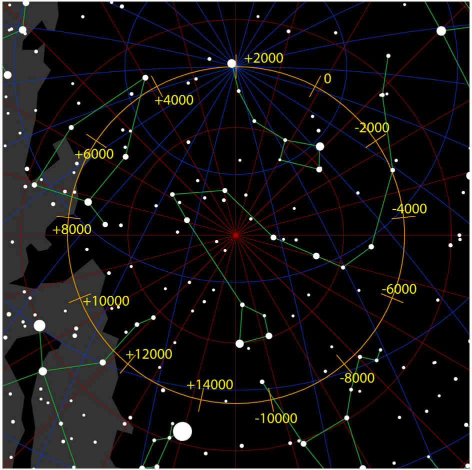
The Earth is not a perfect sphere, but has an equatorial bulge, and the gravitational force, from the Sun and the Moon, on a non-spherical body, causes the precession.

Precession of Earth’s axis

Precession of Earth’s axis around the south ecliptic pole



Precession of Earth’s axis around the north ecliptic pole



Credits: Widipedia

Mean solar day

A natural way to measure time is based on the position of the Sun in the sky.

A (solar) day is defined by the synodic rotation period, that is the synodic day – the time it takes for the Sun to pass over the same meridian.

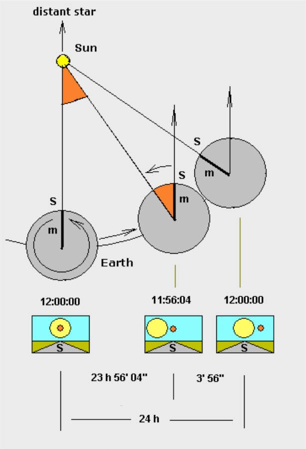
The apparent solar day is the time between two consecutive noons. It is not constant (i.e. $\neq$ 24h) because of

1. the eccentricity of Earth's orbit (i.e. its the non-circularity)
2. the obliquity of Earth (i.e. its axial tilt, the angle between the perpendicular of plane of the orbit and the axis of rotation)

$\rightarrow$ this gives rise to the equation of time.

The mean solar time is the averaged duration of the apparent solar day over 1 yr.

Mean solar day and sidereal day



Credits: Wikipedia, F.J.B. González

$T_0 = 0$h: A distant star (the small red circle) and the Sun are at culmination, on the local astronomical meridian.

$T_1 = 1$ sidereal day: the distant star is again at culmination.

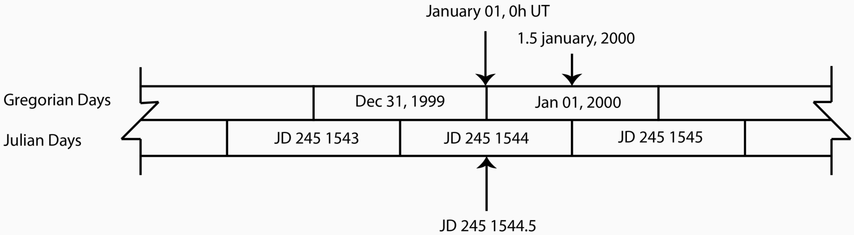
$T_2 = 1$ solar day: few minutes later the Sun is on the local astronomical meridian again at culmination.

The sidereal day is the time it takes for the Earth to make one full rotation with respect to the stars.

The mean solar day is the time it takes for the Earth to make one full rotation with respect to the mean Sun.

The duration of the mean solar day is 24 hours, but the duration of the sidereal day is about 4 minutes less.

Gregorian days vs. Julian days



Julian day is used in the Julian date (JD) system of time measurement for scientific use by the astronomy community, presenting the interval of time in days and fractions of a day since 01 JAN 4713 BC Greenwich noon.

Julian date is recommended for astronomical use by the International Astronomical Union.

Julian days are counted as integers continuously until the present time. This makes it very easy to compare relative times of events and do arithmetic.

Modified Julian days

A disadvantage of Julian days is that the number of days elapsed since the original date is large.

Variants of the Julian day have therefore been defined, such as the Modified Julian Day (MJD), which shifts the original date to 17 NOV 1858 at 00:00.

The conversion between JD and MJD is given by:

$$MJD = JD - 2\,400\,000.5$$

Date (UTC)	MJD	JD
2020-01-01 00:00:00	58 848.5	2 458 849.0
2021-01-01 12:00:00	59 215.0	2 459 215.5

Time references

Standard	Name	Definition
UTC	Coordinated Universal Time	Based on TAI but with leap seconds added to stay within ± 0.9 s of mean solar day
TAI	International Atomic Time	Weighted average of atomic clocks based on the definition of the second
UT1	Universal Time	Based on Earth's rotation
LT	Local Time	Legal local time, usually UTC + time zone
MET	Mission Elapsed Time	Elapsed time since launch of spacecraft

An epoch is a moment in time used as a reference or measurement point.

Orbital motion and Kepler's laws

The two-body problem

- The two-body problem is to determine the motion of the two bodies that interact only with each other.

- Common examples include a satellite orbiting a planet, a planet orbiting a star, two stars orbiting each other (binary star).

Credits: NASA

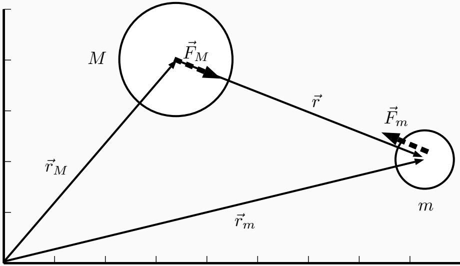
Relative motion in the 2-body problem

$$\vec{F}_m = -G\frac{mM}{||\vec{r}||^3}\vec{r}$$

where we denote $\vec{r} = \vec{r}_m - \vec{r}_M$.

The force on mass M due to mass m is

$$\vec{F}_M = G\frac{Mm}{||\vec{r}||^3}\vec{r} = -\vec{F}_m$$



This is an ordinary differential equation with 12 equations, $\vec{r}_m$ (3 degrees of freedom - dof), $\vec{r}_M$ (3 dof), $\dot{\vec{r}}_M$ (3 dof), $\dot{\vec{r}}_M$ (3 dof). The nonlinearity comes from the $1/||\vec{r}||^3$ term.
$\rightarrow$ consider the relative motion with respect to largest object M to reduce the dof:

$$\ddot{\vec{r}} = \ddot{\vec{r}}_m - \ddot{\vec{r}}_M = -\frac{\vec{F}_m}{m} + \frac{\vec{F}_m}{M} = -\frac{m+M}{mM}G\frac{mM}{||\vec{r}||^3}\vec{r} = G\frac{m+M}{||\vec{r}||^3}\vec{r}$$

$$\implies \vec{F}_m = m\ddot{\vec{r}} = -G\frac{(m+M)m}{||\vec{r}||^3}\vec{r}$$

Recovering the absolute position

How to recover the absolute position? $\rightarrow$ Use a coordinate system centred at centre of mass (barycentre) such that

$$\vec{r}_{\text{CoM}} = \frac{\vec{r}_m m + \vec{r}_M}{m + M}$$

Then we can recover $\vec{r}_m$ from $\vec{r}_{\text{CoM}}$

$$\vec{r}_m = -\frac{M}{m+M}\vec{r}$$

If M is the most massive body (e.g. Earth) and m is a satellite (i.e. negligible), then

$$\frac{m}{m+M} \approx 0 \quad \text{and} \quad \frac{M}{m+M} \approx 1$$

$\rightarrow$ the magnitude of the motion of the Earth due to the Moon is

$$\frac{m_{\text{Moon}}}{m_{\text{Moon}} + m_{\oplus}} r_{\text{Moon-}\oplus} \approx 4652 \text{ km}$$

$\rightarrow$ the Earth-Moon centre of mass (4652 km from Earth's centre) lies beneath the surface of the Earth ($R_{\oplus} = 6378$ km)

The N-body problem

There are many situations where the are more than two bodies (e.g. solar system, the Earth-Moon system, the Milky Way, . . .)

In this case, the force on mass i due to all other masses is

$$m_i \ddot{\vec{r}}_i = \vec{F}_i = G \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{m_i m_j}{||\vec{r}_{ij}||^3} \vec{r}_{ij}$$

The centre of mass of a collection of point masses m_i is

$$\vec{r}_{\mathrm{CoM}} = \frac{1}{\sum_i m_i} \sum_{i=1}^{N} m_i \vec{r}_i$$

The gravitational constant G

The computation of G is not easy, but it allows to easily compute the mass of any object.

$\rightarrow$ how to determine G?

In principle, if you know the mass of two objects (m_1, m_2), you can measure the force F and then $G = \frac{Fr^2}{m_1 m_2}$

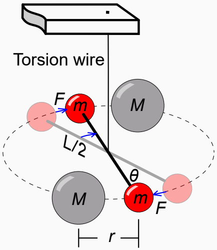
But if m are 1 kg and $r = 1$ cm, $F \sim 6.7$ nN $\rightarrow$ very hard to measure!

The Cavendish experiment

The first measurement of G was made by Cavendish in 1798.

- Suspend two small spheres of mass m, separated by length L
- Move two large masses M within a known and small distance r
- Gravity will create a torque
- Measure the torsion (using e.g. the deflection of a light source by a mirror)

$\rightarrow$ Compute G (for details look up e.g. Wikipedia)



Credits: Wikipedia, C. Burks

- G is not well known (about 22 ppm in 2022) and there are questions about whether G is actually varying with time, $G(t)$?

Newton's law and hypotheses for the rest of the course

$\rightarrow$ Let's go back to an easier problem.

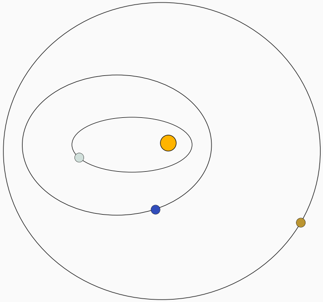
Hypotheses. We will now consider:

- Central body of mass M + spacecraft only *(we will relax this later)*.
- Mass of spacecraft $m \ll M$ the mass of the central body.
- Bodies are spherical and homogeneous *(we will relax this later)*.
- No perturbations *(we will relax this later)*.

$$\boxed{F = G\frac{Mm}{r^2} = \frac{\mu m}{r^2}}$$

Kepler's laws (1609-1619) – First law

The orbit of every planet is an ellipse with the Sun at one of the two foci.

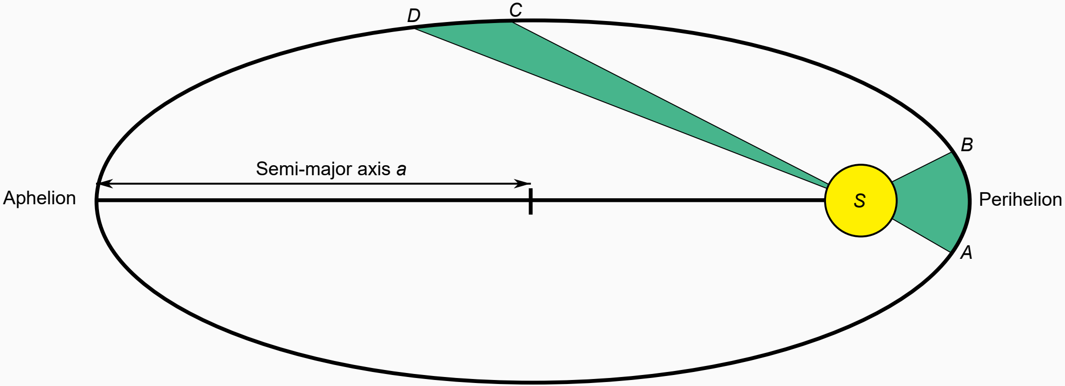


Kepler's laws were established at the beginning of the 17th century from observations of the motion of Mars in the sky made by Tycho Brahe.

The first Kepler's law can be generalised in the case of a two-body problem: the orbit of the small body versus the large body is, without perturbations, a conic, i.e. a circle, an ellipse, a parabola, or hyperbola.

Kepler's laws (1609-1619) – Second law

A line joining a planet and the Sun sweeps equal areas during equal intervals of time.



The conservation of angular momentum implies that if areas *SAB* and *SCD* are equal, the time to go from *A* to *B* is equal to the time to go from *C* to *D*.

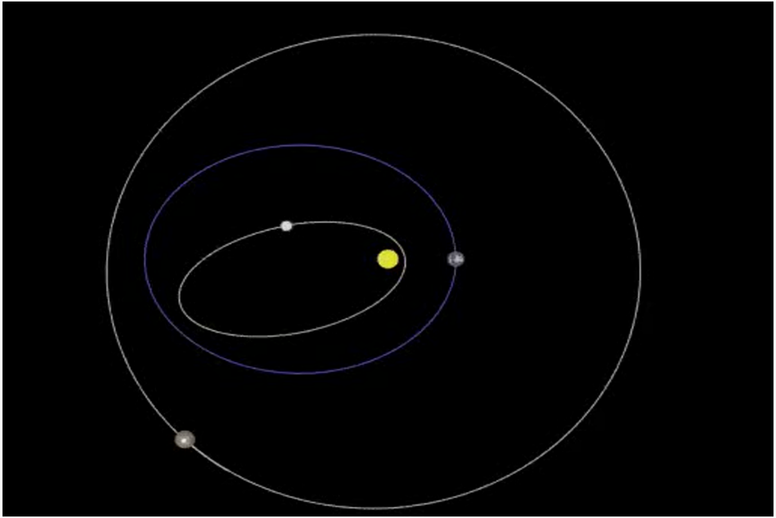
$\Longrightarrow$ The orbital velocity is the highest at perihelion, and the lowest at aphelion

Kepler's laws (1609-1619) – Third law

The square of the orbital period of a planet is proportional to the cube of the semi-major axis of the orbit.

$$T^2 \sim a^3 \implies \boxed{T = 2\pi\sqrt{\frac{a^3}{\mu}}}$$

Kepler's laws (1609-1619) – Summary



Credits: Animations for Physics and Astronomy Education

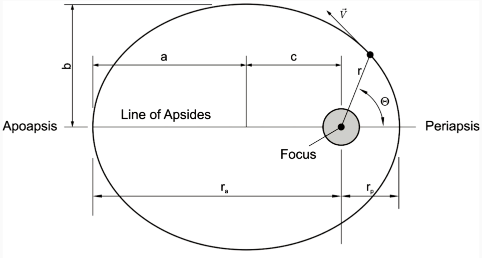
Orbits

As the angular momentum (per unit of mass) $\vec{h} = \vec{r} \times \vec{V}$ is constant over time (i.e. $\dot{\vec{h}} = 0$), the orbits in a two-body system remain in the same plane.

From the equation of relative motion,

$$\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}$$

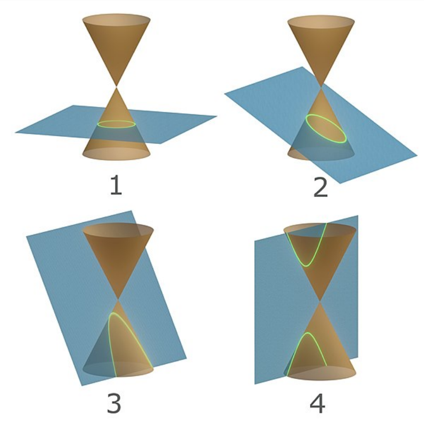
we can integrate to get a scalar equation that describes the motion. (Derivation: $\ddot{\vec{r}} \times \vec{h}$ and integrate to get r)



$$\Longrightarrow \quad r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\Theta}$$

where Θ is the true anomaly (sometimes written as ν) and e is the eccentricity.
→ This describes a trajectory based on a conic section. → 1st Kepler's law

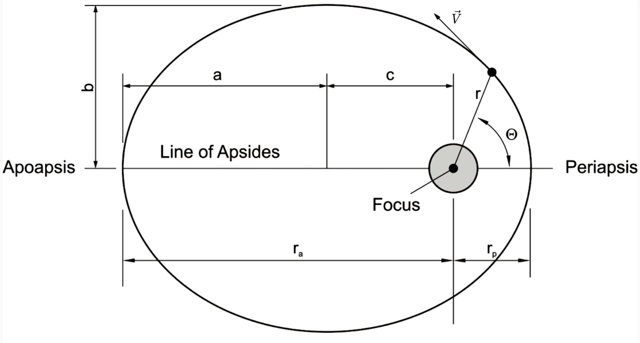
Conic sections



Credits: Wikipedia, JensVyff

1. Circles, $e = 0$
2. Ellipses, $0 < e < 1$
3. Parabolas $e = 1$
4. Hyperbolas $e > 1$

Elliptical orbits



- a semi-major axis
- b semi-minor axis
- $c = ae$, with eccentricity $e < 1$
- r the distance object-focus
- r_a distance to apoapsis
- r_p distance to the periapsis
- $\vec{V}$ velocity
- Θ True anomaly

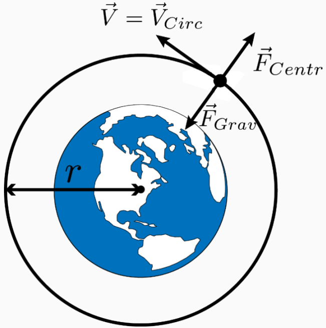
Periapsis and apoapsis are general terms (also pericenter and apocenter). Periastris and apoastris sometimes used for a star as central body. If the Earth is the central body, we talk about perigee and apogee; if it is the Sun, perihelion and aphelion. For the Moon, the suffix is -lune.

The true anomaly is the angle between the direction of the periapsis and the radius vector to the object.

Orbital period

Velocity on a circular orbit is

$$F_{\text{Centr}} = F_{\text{grav}} \implies \frac{V^2}{r} = \frac{\mu}{r^2} \implies \boxed{V = \sqrt{\frac{\mu}{r}}}$$



The period for a circular orbit is

$$\boxed{T = \frac{2\pi r}{V} = 2\pi\sqrt{\frac{r^3}{\mu}}}$$

Orbits are very rarely circular, very often elliptical

$$\boxed{T = 2\pi\sqrt{\frac{a^3}{\mu}}}$$

Energy of the orbital motion

Energy of the orbital motion, per unit mass

$$\boxed{\epsilon = \frac{V^2}{2} - \frac{\mu}{r}}$$

The specific energy in terms of the semi-major axis a is

$$\boxed{\epsilon = -\frac{\mu}{2a}}$$

It depends only on the semi-major axis a.

- Total energy is the sum of the kinetic energy and the potential energy.
- If $V < V_{\text{Esc}}$, which is the case for a closed orbit, elliptical or circular, $\epsilon < 0$ (bound orbit).
- In case of a very elongated ellipse, $\epsilon \lesssim 0$
- In the limit case of a parabolic orbit, $\epsilon = 0$
- If the orbit is hyperbolic, $\epsilon > 0$

Orbital velocity

The orbital velocity at at distance r on an orbit can be derived from the total energy.

For an elliptical orbit,

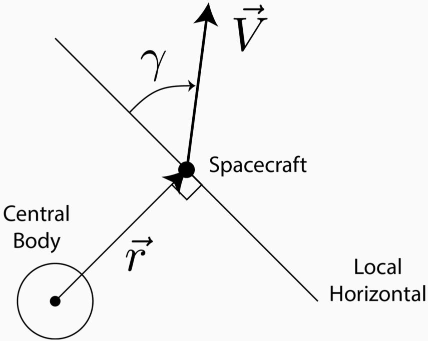
$$\boxed{V = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}} \quad \text{(Vis Viva equation)}$$

(Derivation from the integration of the total energy conservation, i.e. $\int \dot{\epsilon} = \int 0$)

For a circular orbit, this reduces to

$$\boxed{V = \sqrt{\frac{\mu}{r}}}$$

Flight path angle γ



Angular momentum of a spacecraft $\vec{h}$ per unit of mass, again,

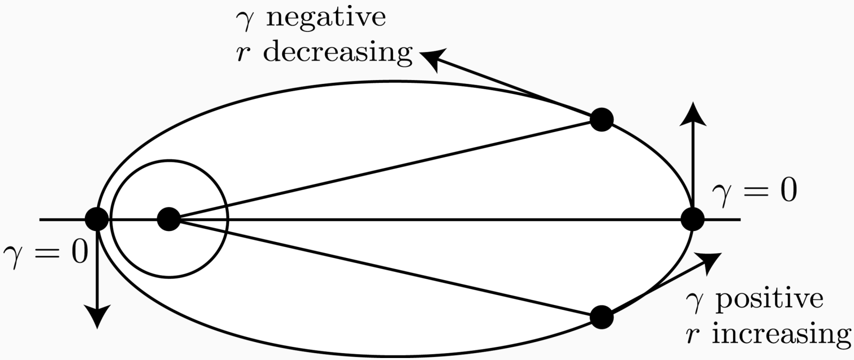
$$\vec{h} = \vec{r} \times \vec{V}$$

$$|\vec{h}| = r \cdot V \cos \gamma$$

where γ is the flight path angle.

The flight path angle is the angle between the direction of the velocity vector and the perpendicular to the radius vector at the location of the spacecraft is.

Variation of the flight path angle γ



The flight path angle is equal to zero at the apogee and perigee (or apoapsis and periapsis). It is positive from the perigee to the apogee and negative from the apogee to the perigee.

Elliptical orbits – Useful formulas

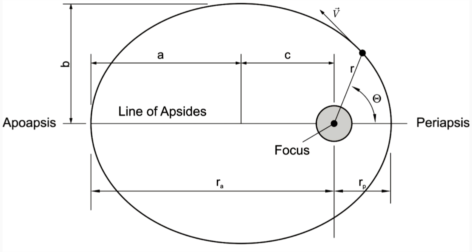
Eccentricity $e = \frac{c}{a} = \frac{r_a - r_p}{r_a + r_p} = \frac{r_a}{a} - 1 = 1 - \frac{r_p}{a} = \frac{r_2 - r_1}{r_1 \cos\theta_1 - r_2 \cos\theta_2}$

Flight path angle $\tan\gamma = \frac{e \sin\theta}{1 + e\cos\theta}$

Mean motion $n = \sqrt{\frac{\mu}{a^3}}$

Period $T = \frac{2\pi}{n} = 2\pi\sqrt{\frac{a^3}{\mu}}$

Radius $r = \frac{a(1-e^2)}{1+e\cos\theta} = \frac{r_p(1+e)}{1+e\cos\theta}$



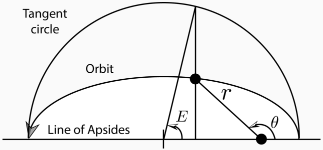
Elliptical orbits – Useful formulas

Apoapsis radius $r_a = a(1+e) = 2a - r_p = r_p \frac{1+e}{1-e}$

Periapsis radius $r_p = a(1-e) = 2a - r_a = r_p \frac{1-e}{1+e} = \frac{r_1(1+e\cos\theta_1)}{1+e}$

Semi-major axis $a = \frac{r_a + r_p}{2} = \frac{r_p}{1-e} = \frac{r_a}{1+e}$

True anomaly $\cos\theta = \frac{r_p(1+e)}{re} - \frac{1}{e} = \frac{a(1-e^2)}{re} - \frac{1}{e}$



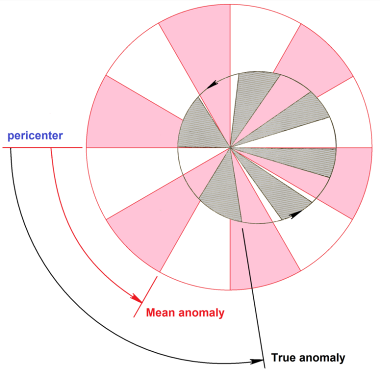
Eccentric anomaly $\cos E = \frac{e+\cos\theta}{1+e\cos\theta}$

Velocity $V = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$ $\quad r_p V_p = r_a V_a$

Mean and ecentric anomaly

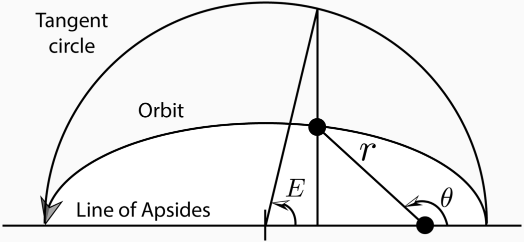
The mean anomaly M is the mean fraction of orbit elapsed since passed periapsis

$$M = nt$$



Credits: Wikipedia, Tfr000

E, the eccentric anomaly, is an angle that defines the position on a tangent circle of an object that is moving along an elliptical orbit.



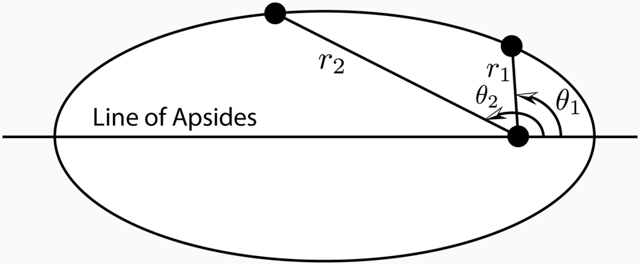
Kepler's equation is transcendental. It cannot be solved for E but expresses the time evolution of E since passing the periapsis:

$$t = \frac{E - e \sin E}{n}$$

Elliptical orbits - Example

Determination of parameters of an elliptical orbit knowing two points on the orbit.

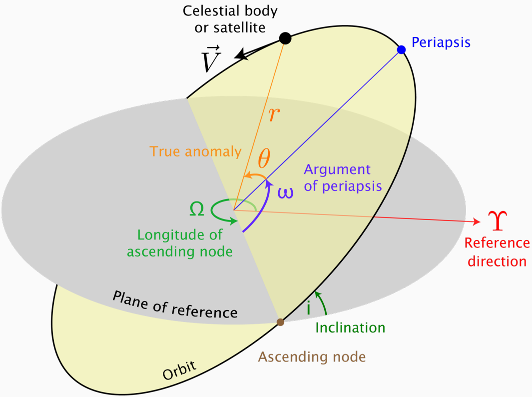
$$r_i = \frac{r_p(1+e)}{1+e\cos\theta_i}$$



$$e = \frac{c}{a} = \frac{r_2 - r_1}{r_1\cos\theta_1 - r_2\cos\theta_2} \qquad r_p = \frac{r_1(1+e\cos\theta_1)}{1+e} \qquad r_a = r_p\frac{1+e}{1-e}$$

not quite orbit determination, see later discussion.

Classical orbital elements



e, *a*, θ already defined

i inclination of the orbital plane

Ω longitude or Right Ascension of the Ascending Node (RAAN) in the plane of reference

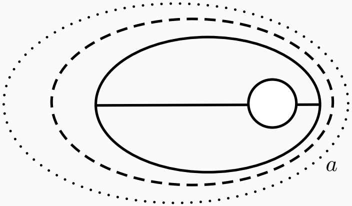
ω argument of periapsis

Current time *t* allowing a determination of the exact position of the object.

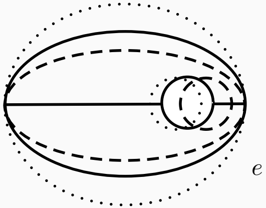
Credits: Adapted from Wikipedia, Lasunnkty

The spacecraft is passing from the southern celestial hemisphere to the northern on a point on the plane of reference called the ascending node. The descending node is on the other side, when going from N to S.

Orbit size: semi-major axis *a* and eccentricity *e*

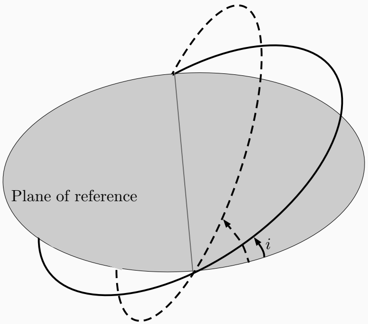


The semi-major axis *a* describes the size of the orbit.

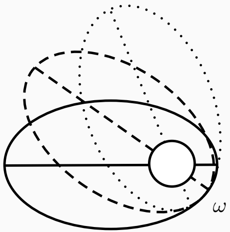


The eccentricity *e* describes the “roundness” of the orbit.

Orbit orientation: inclination *i* and argument of perigee ω

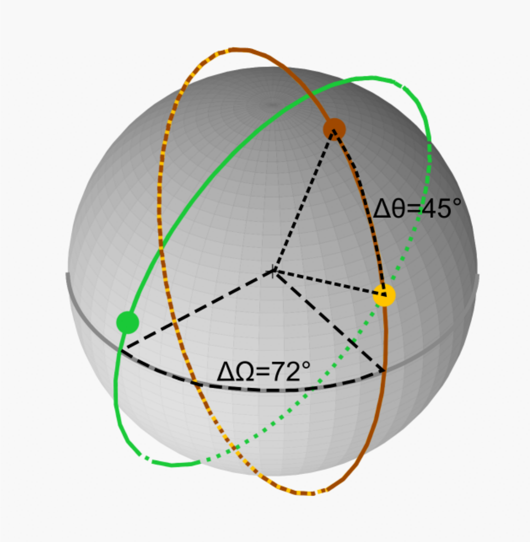


The inclination *i* is the tilt of the orbit.



The argument of perigee ω is the angle between the direction of the ascending node and the direction of the perigee.

Orbit orientation: RAAN Ω and true anomaly θ



The sketch shows three satellites that have the same a, e, i, ω.

The difference in RAAN $\Delta\Omega$ between the green and brown/yellow orbits is 72°.

Yellow and brown are on the same orbital plane, but separated by 45°. The true anomaly θ is measured from the perigee.

State vector $\leftrightarrow$ Orbital elements

To define fully the position in 3 dimensions in the geocentric-inertial (or heliocentric-inertial) coordinate system, you need

$$(X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}, t)$$

that is the position vector, the velocity vector and an epoch.

$$\Updownarrow$$

A formulation of the state vector in terms of classical orbital elements is equivalent

$$(a, e, i, \Omega, \omega, \theta, t)$$

$\rightarrow$ 6 parameters describing the orbit + time are required to write the state vector

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