Solution to Exercise Session 3

Note Problems marked with a ($\star$) are complimentary exercises and will not be solved in class.

Problem 1 Multiple Choice Questions

A) ($\star$) *A spacecraft is on a parking orbit at 1000 km above the surface of the Earth. At some point, the spacecraft is accelerated to a departure velocity of 11 km/s. What will be the speed on the surface of the sphere of influence and at infinity?*

(1) 0.32, 0 km/s
(2) 1.18, 1.12 km/s
(3) 3.72, 3.60 km/s
(4) 6.15, 3.92 km/s

When the spacecraft reaches the surface of the sphere of influence, it is on a hyperbolic trajectory. Using the conservation of energy, we can write:

$$\frac{v_S^2}{2} - \frac{\mu_\oplus}{r_S} = \frac{v_d^2}{2} - \frac{\mu_\oplus}{r_d}$$

where v_d is the departure velocity, r_d is the radius of the parking orbit, v_S is the speed at the sphere of influence (SOI) and r_S is the radius of the SOI. The radius of the SOI is: $r_S = d_{\odot\oplus}\left(\frac{\mu_\oplus}{\mu_\odot}\right)^{\frac{2}{5}} = 0.924 \cdot 10^6$ km. Solving for v_S gives: $\mathbf{v_S = 3.72\ km/s}$.

The speed at infinity, or departure hyperbolic excess velocity v_d^∞, is the value of v_S such that

$$\frac{v_S^2}{2} - \frac{\mu_\oplus}{r_S} = \frac{v_d^2}{2} - \frac{\mu_\oplus}{r_d} \qquad \text{with } r_S \to \infty$$

We can write $\lim\limits_{r_s \to +\infty} v_S = v_d^\infty$ Thus $(v_d^\infty)^2 = v_d^2 - v_{\mathrm{Erd}}^2$ where $v_{\mathrm{Erd}} = \sqrt{\frac{2\mu_\oplus}{r_d}}$ is the escape velocity from a distance r_d of the centre of the Earth. Thus, $\mathbf{v_d^\infty = 3.60\ km/s}$. This exercise shows that the sphere of influence is a good approximation of "infinity" for our purpose.

B) *The MSL spacecraft, which carried Curiosity to Mars, had a mass of 4050 kg at 200 km altitude on a LEO parking orbit. Its orbital velocity was 7.78 km/s. To reach Mars, it had to accelerate to a departure velocity v_d of 11.50 km/s with a specific impulse of 320 s at a mass expulsion rate of 5 kg/s. What was the duration of the burn?*

(1) 20 s
(2) 562 s
(3) 1832 s

(4) 102 s

From Tsiolkovsky rocket equation we have:

$$\Delta v = g_0 I_{\text{sp}} \ln\left(\frac{m}{m - \dot{m}t}\right)$$

where m is the mass of the satellite, $\dot{m}$ is the mass flow and t_b the duration of the burn. Therefore:

$$t_b = \frac{m\left(\exp\left\{\frac{\Delta v}{g_0 I_{\text{sp}}}\right\} - 1\right)}{\dot{m}\exp\left\{\frac{\Delta v}{g_0 I_{\text{sp}}}\right\}} = \mathbf{562\ s} \tag{1}$$

C) *($\star$) A similar spacecraft has ion thrusters of specific impulse 3200 s. We want to perform a similar increase of velocity. Would the "burn" be shorter or longer?*

(1) Longer
(2) Shorter

Why?

(1) The I_{sp} is much higher
(2) The thrust is much lower
(3) The exhaust velocity is much lower
(4) The thrust is much higher

The Ion thrusters have a much lower thrust typically $\sim 10^{-2}$ N. If we use newton's second law, we have

$$F = m\frac{\Delta v}{\Delta t}$$
$$\Delta t = \frac{m\Delta v}{F}$$

It means that for the same Δv as before, the burn time needs to be longer since we have a lower thrust. Electric propulsion is not used for short powerful burns, but is used for maneuvers carried out very gradually.

D) ($\star$) *A rocket is in free space, made of propellant only. Its initial mass is m and its thrust constant at a value mg_0. Its specific impulse is I_{sp}. It burns off completely in a time Δt. What is the value of Δt ?*

(1) Its $\mathbf{I_{sp}}$ in seconds
(2) 9.81 s
(3) 42 s
(4) Cannot be determined with available data.

The time Δt to consumes a mass m_p of propellant is given by:

$$\Delta t = \frac{m_p}{\dot{m}_p}$$

As the mass of the rocket m is equal to the mass of propellant m_p, the time Δt can be derived from the definition of the specific impulse:

$$\Delta t = \frac{m}{\dot{m}_p} = \frac{mg_0}{\dot{m}_p g_0} = \frac{F}{\dot{m}_p g_0} = I_{sp}$$

And therefore the time needed to burn off completely this rocket full of propellant is its specific impulse in second.

E) *The Galileo mission which explored the Jovian system used the gravity assist of Venus during its complicated trip to Jupiter's neighbourhood. Its* $\boldsymbol{v_a^\infty}$ *at the entrance of the sphere of influence of Venus was perpendicular to the velocity vector of the planet and had an amplitude of 6.2 km/s. The gravity assist manoeuvre resulted in a deflection of Galileo's planetocentric velocity vector by* $\delta = +43°$*. The heliocentric arrival velocity at Venus was* $V_{before} = 37.4$ *km/s. The heliocentric velocity of Venus is* $V_v = 35.2$ *km/s. Estimate the heliocentric velocity of the spacecraft at the departure from the sphere of influence of Venus.*
Hint. *Use the law of the cosine* $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$ *to compute the heliocentric velocity after the slingshot.*

(1) Cannot be determined with available data.
(2) 33.7 km/s
(3) 35.2 km/s
(4) 39.7 km/s

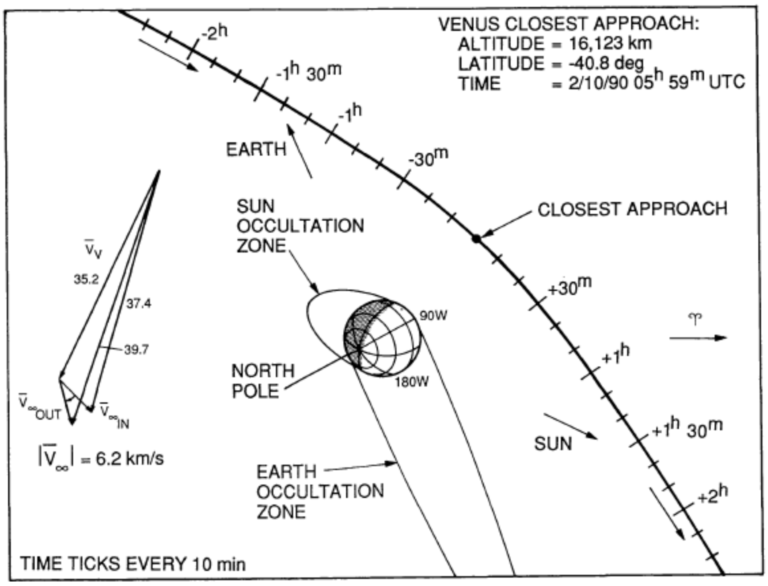


Figure 1: *Pole view of the Venus flyby on February 10, 1990 – Reproduction of D'Amario (1992) – http://adsabs.harvard.edu/abs/1992SSRv...60...23D.*

The important point to realise here is that the arrival and the departure velocities in the reference frame of the planet have the same *amplitude* of $v_d^\infty = v_a^\infty = 6.2$ km/s. The flyby will only change the *direction* of the velocity vector of the space probe ($\boldsymbol{v_d^\infty} \neq \boldsymbol{v_a^\infty}$). Hence, the speed gained in the reference frame of the Sun is gained by a different additions of Venus velocity vector V_v and v^∞. Using the cosine law with $\gamma = 90° + 43°$, $a = V_v$, $b = v_d^\infty = v_a^\infty$, $c = V_{\text{after}}$ we get:

$$V_{\text{after}}^2 = V_v^2 + (v_a^\infty)^2 - 2V_v v_a^\infty \cos\gamma \implies \mathbf{V}_{\text{after}} = \mathbf{39.7\ km/s}$$

This corresponds to a Δv of $\Delta v = 39.7 - 37.4 = 2.3\,\text{km/s}$.

Actually, the angle between the arrival infinity velocity and the velocity vector of Venus was not exactly 90° and δ not 43°(see Fig. 1), but that is a good enough approximation.

Problem 2 Interplanetary Transfer

We plan the interplanetary transfer of a spacecraft from the Earth to Jupiter. We assume that both orbits are circular and coplanar for simplicity.

A) *Assume that the spacecraft follows a Hohmann transfer trajectory. Determine the departure hyperbolic excess velocity v_d^∞ needed to leave the influence of the Earth on our way to Jupiter and the arrival hyperbolic excess velocity v_a^∞ .*

B) *What is the departure velocity v_d to achieve this mission from a circular parking orbit around the Earth at 200 km altitude?*

C) *Find the spacecraft's energy (per unit mass) on the hyperbolic Earth escape trajectory.*

D) *What is the time needed for the trip from Earth to Jupiter, on an Hohmann transfer trajectory?*

E) *Upon arrival in the sphere of influence of Jupiter, the impact parameter is chosen so as to achieve, in the hyperbolic trajectory inside the Jupiter sphere of influence, a closest distance to the center of Jupiter equal to 100,000 km. What is the value of the braking impulse Δv_i so as to inject the spacecraft on a circular orbit around Jupiter at that altitude?*

F) *What is the phase angle between the Earth and Jupiter at Earth's departure?*

G) *How often does this launch windows occur?*

Numerical values:
Radius of Jupiter's orbit $R_{\text{Jupiter}} = 5.204$ AU.
Mass of Jupiter $m_{\text{Jupiter}} = 1.8989 \cdot 10^{27}$ kg, mean radius $r_{\text{J}} = 69,911$ km.
Solution.

A) Outside the sphere of influence of the Earth and of Jupiter, the trajectory is a simple Hohmann transfer with the Sun at one focus of the elliptical transfer orbit, Earth at the perigee and Jupiter at the apogee. The heliocentric velocity of the spacecraft at the departure is: (Remember capital V means with respect to the Sun!)

$$V_{d,\text{S/C}} = \sqrt{\frac{2\mu_\odot}{R_\oplus} - \frac{\mu_\odot}{a}}$$

where $a = (R_\oplus + R_{\text{Jupiter}})/2$. Therefore, the departure hyperbolic excess velocity needed to leave the sphere of influence of our planet is:

$$v_d^\infty = V_{d,\text{S/C}} - V_\oplus = \sqrt{\frac{2\mu_\odot}{R_\oplus} - \frac{\mu_\odot}{a}} - \sqrt{\frac{\mu_\odot}{R_\oplus}} = \mathbf{8.8\ km/s}$$

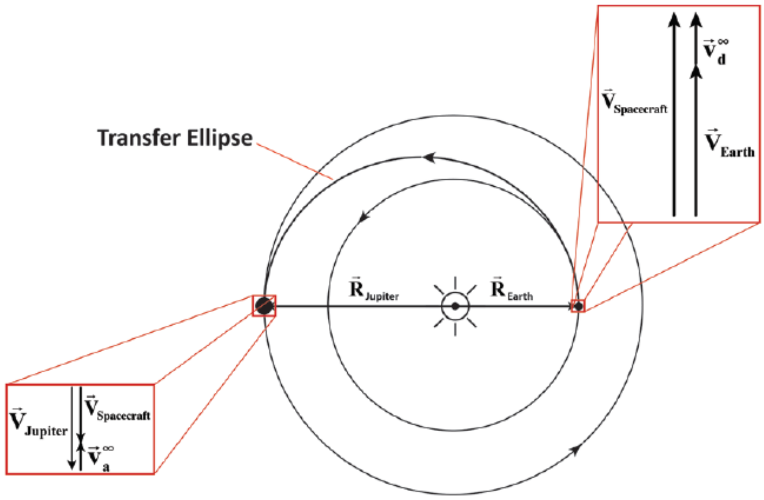
Upon arrival, the heliocentric velocity is described by:

$$V_{a,\text{S/C}} = \sqrt{\frac{2\mu_\odot}{R_{\text{Jupiter}}} - \frac{\mu_\odot}{a}}$$

Thus, the arrival hyperbolic excess velocity at the sphere of influence of Jupiter is:

$$v_a^\infty = V_{a,\text{S/C}} - V_{\text{Jupiter}} = \sqrt{\frac{2\mu_\odot}{R_{\text{Jupiter}}} - \frac{\mu_\odot}{a}} - \sqrt{\frac{\mu_\odot}{R_{\text{Jupiter}}}} = \mathbf{-5.6\ km/s}$$

The negative sign indicates that we arrive at Jupiter with a smaller heliocentric velocity than Jupiter itself.



B) The departure velocity to achieve this mission is given by:

$$v_d^2 = (v_d^{\infty})^2 + v_{\text{Erd}}^2$$

where v_{Erd} is the escape velocity at the distance r_d from the Earth's center:

$$v_{\text{Erd}} = \sqrt{\frac{2\mu_{\oplus}}{r_d}} = 11.0 \text{ km/s}$$

hence yielding $v_d = \mathbf{14.1 \text{ km/s}}$.

C) The orbital energy in a hyperbolic orbit is $\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$. This energy in a hyperbolic geocentric escape trajectory where $r \to \infty$ would be $\varepsilon = \frac{(v_d^{\infty})^2}{2} = \mathbf{38.6 \text{ km}^2/\text{s}^2}$ (= MJ/kg). The trajectory has an energy > 0 and therefore the spacecraft is not bound to the Earth.

D) The travel time is half the period which is given by:

$$t_{tr} = T/2 = \pi\sqrt{\frac{a^3}{\mu_{\odot}}} = \mathbf{2.73 \text{ yr}} = 998 \text{ days}$$

E) The arrival velocity at the periapsis of the hyperbolic orbit inside Jupiter's sphere of influence can be derived from

$$v_p^2 = v_{\text{Esc}}^2 + (v_a^{\infty})^2$$

Where, v_{Esc} is the escape velocity from Jupiter at 100,000 km from the center of the planet:

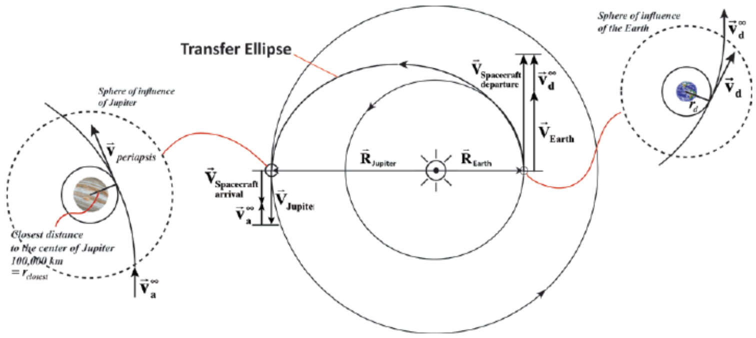
$$v_{\text{Esc}} = \sqrt{\frac{2\mu_J}{r_{\text{closest}}}} = 50.3 \text{ km/s}$$

hence $v_p = 50.6$ km/s. The orbital velocity in a circular orbit of radius r_{closest} is

$$v_{\circ, r_{\text{closest}}} = \sqrt{\frac{\mu_J}{r_{\text{closest}}}} = 35.6 \text{ km/s}$$

Hence the value of the braking impulse must be

$$\Delta v_{\text{insertion}} = v_{\circ,r_{\text{closest}}} - v_p = \mathbf{-15\ km/s}$$



F) As we assume that the planetary motions are circular (see Fig. 2), we have

$$\phi_0 = \pi - n_{\text{Jupiter}} \cdot t_{tr} = \mathbf{97.16}^\circ$$

where the mean motion is given by $n_{\text{Jupiter}} = \sqrt{\frac{\mu_\odot}{a^3_{\text{Jupiter}}}}$.

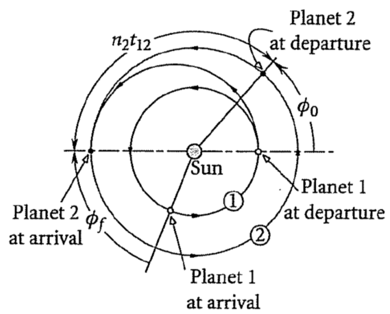


Figure 2: *Ideal phase alignment for an interplanetary transfer*

G) This specific configuration happens at given intervals which are given by the synodic period :

$$T_{syn} = \frac{T_\oplus \cdot T_{\text{Jupiter}}}{|T_\oplus - T_{\text{Jupiter}}|} = \mathbf{1.09}\ \text{yr}$$