Solution to Exercise Session 2

Note Problems marked with a ($\star$) are complimentary exercises and will not be solved in class.

Problem 1 Multiple Choice Questions

A) *We perform a rendezvous with the ISS, which orbits the Earth in a circular LEO at an altitude of 400 km. During the approach, the chaser (Space Shuttle) is in an elliptical orbit in the same plane as the ISS with an apogee at 400 km and a perigee at 370 km, behind the ISS. On each successive apogee crossing, will the Shuttle get closer to the ISS or further away? By how many kilometers? (Give your answer as measured with respect to the orbit of the ISS).*

(1) Closer	(1) -282 km	(4) -4.3 km
(2) Further away	(2) 282 km	**(5) 141 km**
	(3) 7.7 km	(6) -152 km

Catch-up rate (per orbit) for the chaser in an elliptical orbit of semi-major axis a and the TGT in a circular orbit of radius r: $\Delta x = 3\pi(r - a)$.

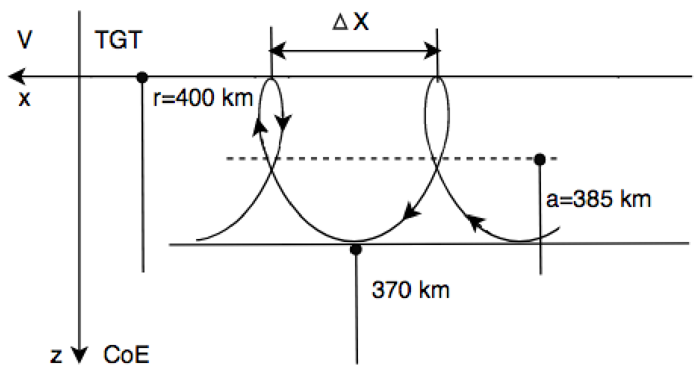


Figure 1: *Relative trajectory of the chaser*

Thus r - a =15 km, and $\Delta x = 3\pi 15\,km = 141\,km$

The chaser (Shuttle) will get closer to the TGT (ISS) on each apogee crossing.

B) *To reach the ISS, the Space Shuttle will have to execute different maneuvers. During rendezvous, a maneuver is performed at apogee, which raises the perigee by 10 km. What is the* Δv *needed for this maneuver?*

(1) 0.22 m/s
(2) 1.43 m/s
(3) 2.86 m/s
(4) 0.32 km/s
(5) 7.11 km/s

We use the LEO approximation $\Delta r \simeq 3.5\Delta v$ with $\Delta r = 10$ km for the perigee altitude change:

$$\Delta v \simeq \frac{\Delta r}{3.5} = \frac{10}{3.5} = \mathbf{2.86} \quad \mathbf{m/s}.$$

C) ($\star$) *CHEOPS was launched in 2019. This exoplanet observation satellite, partly designed at EPFL, is in a Sun-synchronous orbit with an inclination of 98.6° to avoid long eclipses, which allows to reduce battery load. To achieve this strategy of minimising eclipses, what are the local mean solar times when the satellite crosses the Equator?*

(1) Noon/midnight
(2) 3 pm/9 pm
(3) 6 am/6 pm (sunrise/sunset)
(4) 10 am/4 pm
(5) It does not matter.

A satellite on a Sun-synchronous orbit crosses the Equator always at around the same mean solar time. To minimise eclipses, the orbit of CHEOPS is close to the terminator, the day/night boundary on the surface of the Earth, so that the satellite is nearly always illuminated by the Sun. This results in equatorial crossing at about 6 am/6 pm mean solar time (sunrise/sunset).

D) *We want to inject a GPS satellite from a circular parking orbit at 230 km altitude to a final circular orbit at 20'000 km altitude, using a Hohmann transfer without orbital plane change. What are the amounts of the two maneuvers* Δv_1 *and* Δv_2 *?*

(1) 2.1, 3.7 km/s
(2) 1.4, 1.1 km/s
(3) 1.4, 2.1 km/s
(4) 2.1, 1.4 km/s
(5) 4.7, 4.4 km/s

The circular orbital velocities are given by $v = \sqrt{\frac{\mu_\oplus}{R_\oplus + h}}$. Therefore :

$$\begin{aligned} v_{\text{LEO}} &= \sqrt{\frac{\mu_\oplus}{R_\oplus + 230 \text{ km}}} = 7.76 \text{ km/s} \\ v_{\text{GPS}} &= \sqrt{\frac{\mu_\oplus}{R_\oplus + 20\,000 \text{ km}}} = 3.89 \text{ km/s} \end{aligned}$$

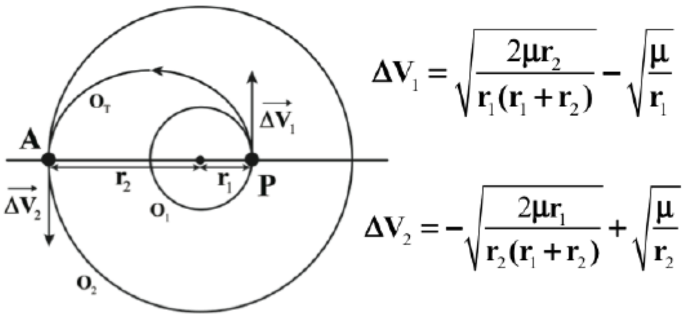


Figure 2: *Hohmann Transfer*

Using Fig. 2, one can compute the Δv required:

$$\begin{aligned}\Delta v_1 &= 2.05 \text{ km/s}\\ \Delta v_2 &= 1.43 \text{ km/s}\\ \Delta v_{\text{tot}} &= \Delta v_1 + \Delta v_2 = 3.48 \text{ km/s}\end{aligned}$$

Problem 2 Hohmann transfer and plane change

A satellite launched from Cape Canaveral (inclination 28.5°) is in a circular low Earth orbit (LEO) at an altitude of 450 km. We want to use the Hohmann transfer technique to raise the altitude to a circular geosynchronous orbit.

A) *What are the values of the two Δvs required for this manoeuvre ? What are the orbital velocities for the initial parking orbit in LEO and for the final geosynchronous orbit ?*

B) *If we want to change to a geostationary orbit, what will be the additional Δv or Δvs ? What is the best strategy for the execution this Δv or Δvs and when ?*

C) *Using the results of the previous questions, what are the values of the Δvs involved ?*

Solution.

A) The altitude of the geosynchronous orbit r_{GEO} is given by $T = 2\pi\sqrt{\frac{r_{\text{GEO}}^3}{\mu_\oplus}}$ where T is the duration of the sidereal day (23h 56min 4.09s) which gives an altitude of $h_{\text{GEO}} = 35'785$ km. The circular orbital velocities are given by $v = \sqrt{\frac{\mu_\oplus}{R_\oplus + h}}$. Therefore :

$$\begin{aligned}v_{\text{LEO}} &= \sqrt{\frac{\mu_\oplus}{R_\oplus + 450 \text{ km}}} = 7.64 \text{ km/s}\\ v_{\text{GEO}} &= \sqrt{\frac{\mu_\oplus}{R_\oplus + 35'785 \text{ km}}} = 3.07 \text{ km/s}\end{aligned}$$

Using Fig. 2, one can compute the Δv required:

$$\begin{aligned}\Delta v_1 &= 2.38 \text{ km/s}\\ \Delta v_2 &= 1.45 \text{ km/s}\\ \Delta v_{\text{tot}} &= \Delta v_1 + \Delta v_2 = 3.83 \text{ km/s}\end{aligned}$$

B) The additional Δv is due to the plane change. The difference between the two orbits, geo*synchronous* and geo*stationary*, is that the latter is on the equatorial plane ($i = 0°$). It would be performed at the apogee of the Hohmann transfer (when the second burn occurs). There are two options :

(1) We can perform the plane change first and then circularize the orbit to a geostationary status or

(2) We can combine the burns for the circularization of the orbit and the plane change to take advantage of their composition law (law of cosines, see next point).

As we want to achieve an orbit with a 0° inclination, the circularization Δv) has to be performed over the equator. Therefore the line of node of the transfer orbit has to be in the equatorial plane. This implies to perform the first boost (Δv_1) over the equator as well.

C) The velocity of the satellite at the apogee of the Hohmann transfer is

$$v_a = \sqrt{\mu_\oplus \left(\frac{2}{r_2} - \frac{1}{a}\right)} = \sqrt{2\mu_\oplus \left(\frac{1}{r_2} - \frac{1}{r_1 + r_2}\right)} = 1.62 \text{ km/s}$$

The plane change alone requires a change of velocity of :

$$\Delta v_{\Delta i} = 2 v_{r_2} \sin(\Delta i/2) \approx 800 \text{ m/s}$$

If we were to do a combined manoeuvre, we have to use the law of cosine. The law of cosines relates the lengths of the sides of a plane triangle to the cosine of one of its angles. The required Δv can be calculated from ($\Delta i = 28.5°$):

$$\Delta v_{\text{combined}}^2 = v_a^2 + v_{\text{GEO}}^2 - 2 v_a v_{\text{GEO}} \cos(\Delta i) \implies \Delta v_{\text{combined}} = 1.82 \text{ km/s}$$

The value of the total Δv carried out if two burns are done is 2.25 km/s where as the if we combine the two this drops to only 1.82 km/s – which represents a reduction of 25% ! See Fig. 3.

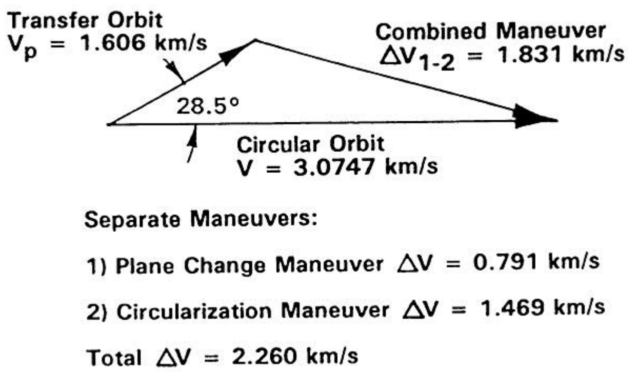


Figure 3: Δv *for a Hohmann transfer and plane change*

Problem 3 ($\star$) Geostationary orbit

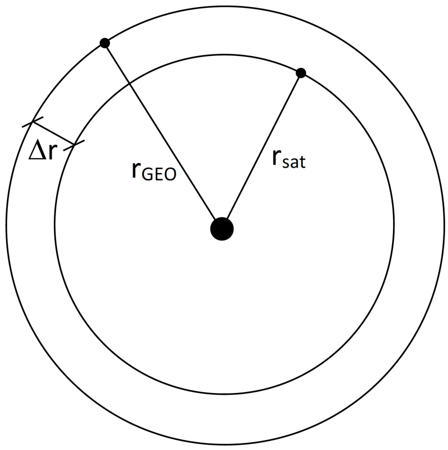
A satellite is launched to a circular geostationary orbit, in the Earth equatorial plane. After receiving the latest position information, the operator realizes that the satellite is orbiting on a circular orbit at an altitude 2km below the target GEO orbit altitude.

A) *Compute the drift in longitude, in °/day, due to the lower altitude.*

B) *What strategy could the operator use to put the satellite back on the correct GEO orbit?*

C) *Given the chosen strategy, compute the total ΔV necessary to reach the target orbit.*

Solution.



We have $r_{GEO} = 42164.2\ km$, $T_{GEO} = 86164\ s$ and $r_{sat} = 42162.2\ km$.

A) As $r_{sat} < r_{GEO}$, the satellite will cover slightly more than an orbit in one sidereal day and thus drift towards the East.

As $\Delta r = r_{GEO} - r_{sat}$ is small, the drift to the East per orbit, versus an object on the GEO orbit, can be approximated by

$$\Delta x \approx 3\pi\Delta r \tag{1}$$

Thus the drift in longitude per orbit will be given by

$$\Delta\text{long/orbit} = \frac{2\pi\Delta x}{2\pi r_{sat}} = \frac{3\pi\Delta r}{r_{sat}} = 0.000447 \text{ rad/day} = \mathbf{0.0256\ ^\circ/day} \tag{2}$$

B) As the difference in radius of the two orbits is small compared to the radii themselves, we can use a **simplified Hohmann transfer** to bring the satellite to the desired GEO orbit.

C) The simplified Hohmann transfer (small ΔV, small Δr) states that

$$\frac{\Delta r}{r} \approx 4\frac{\Delta V}{V} \tag{3}$$

with $\Delta r = 2\ km$, V_{sat}, the speed on the initial orbit, can be computed as follows

$$V_{sat} = \sqrt{\frac{\mu_{\oplus}}{r_{sat}}} = 3074.7\ m/s \tag{4}$$

We obtain for the 1st burn

$$\Delta V = \frac{1}{4}\frac{\Delta r}{r}V = 0.0364\ m/s \tag{5}$$

The simplified Hohmann transfer states that the total ΔV, including the 1st and 2nd burns, can be approximated by $\Delta V_{total} = 2\Delta V$.

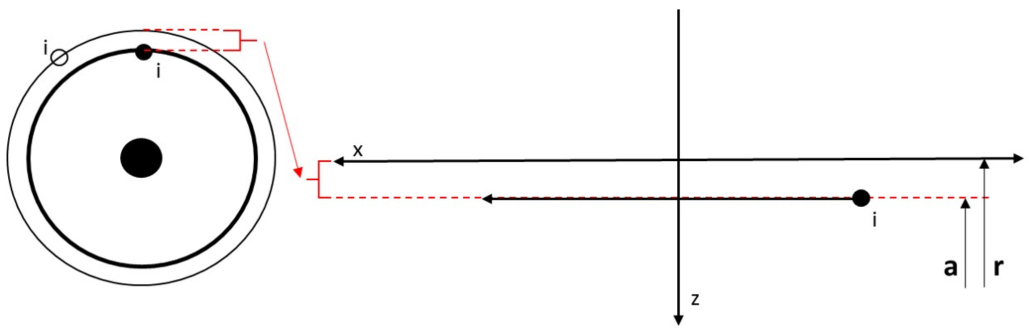
We then have

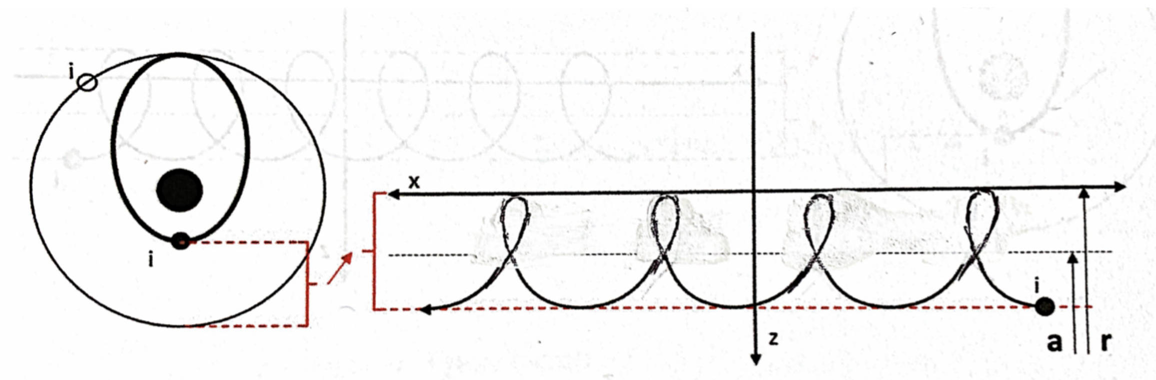
$$\Delta V_{total} = \mathbf{0.0728\ m/s} \tag{6}$$

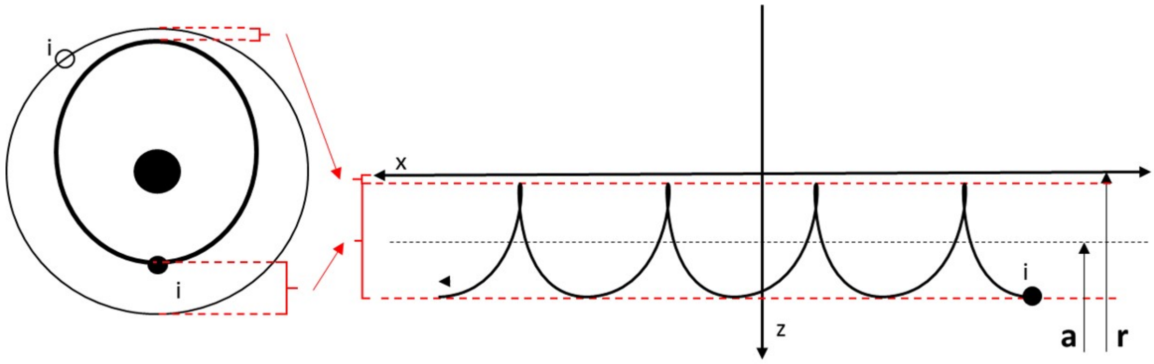
Problem 4 Chaser and Target

For each of the configurations and initial conditions (i) listed below, draw the trajectory of the chaser (thick line orbit) vs. target (thin line orbit). In all cases the direction of motion is counter clockwise, for both the chaser and the target.

Solution.



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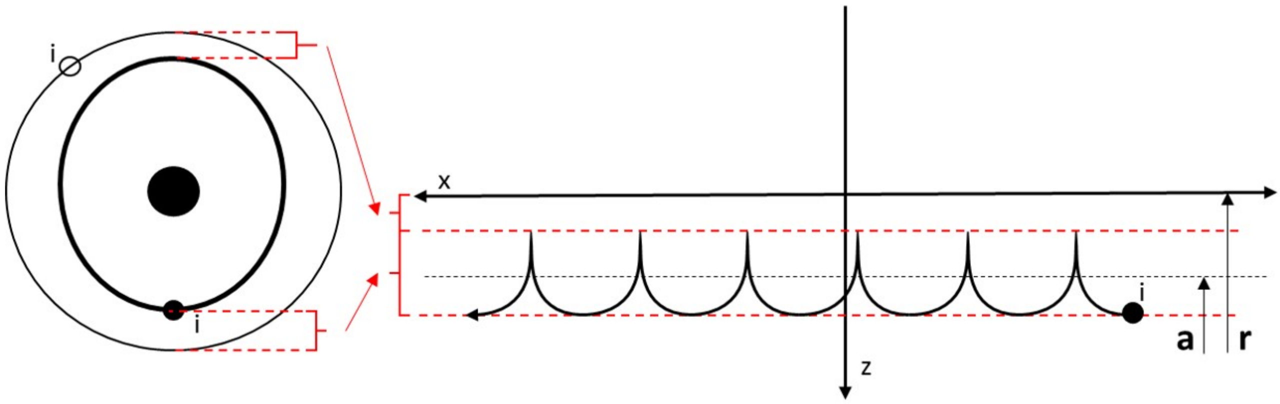
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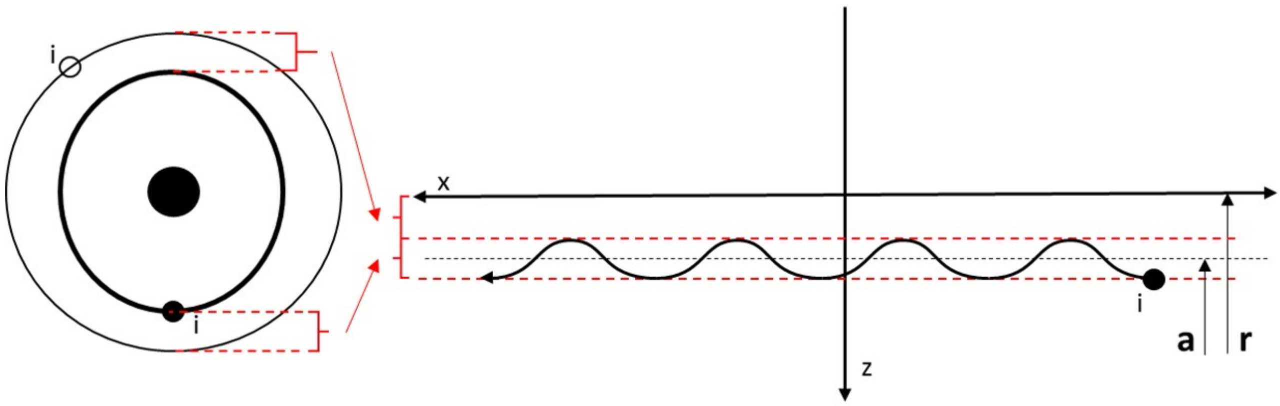
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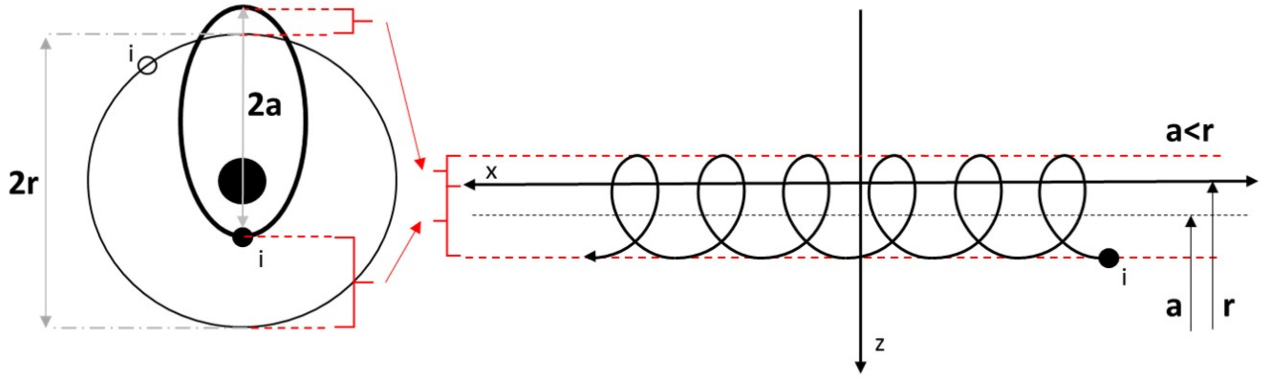
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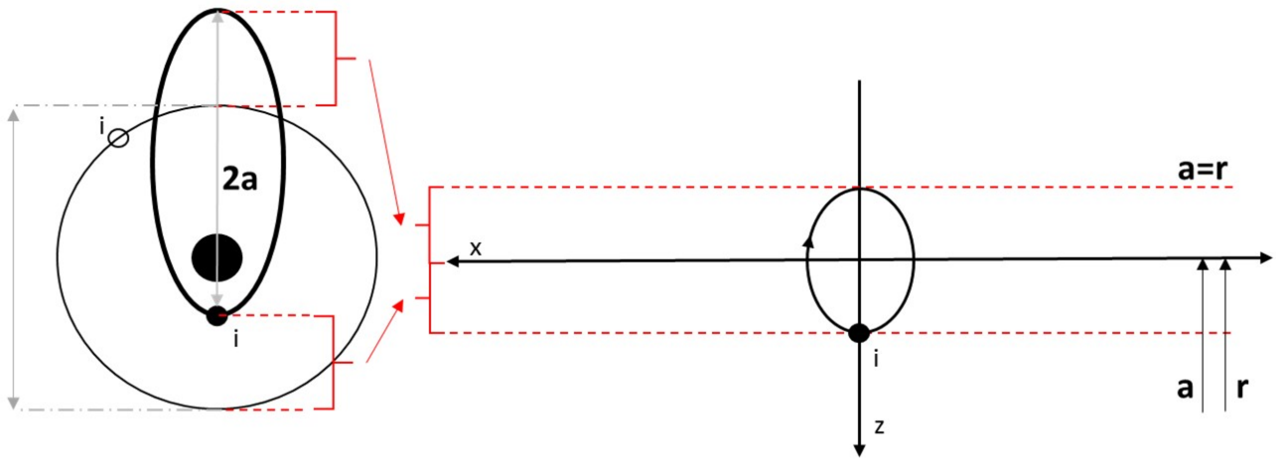
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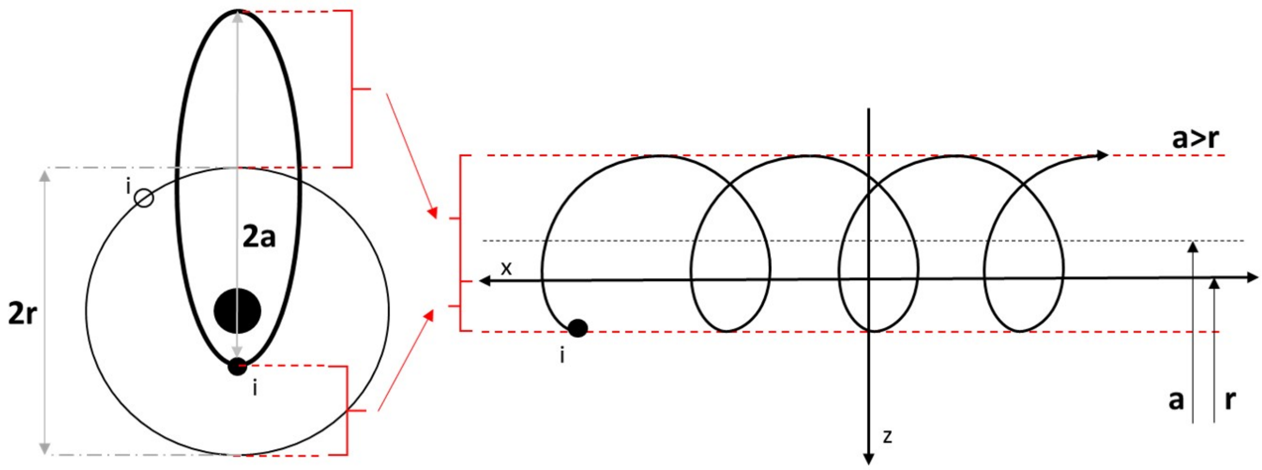
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