Fundamentals of Analog VLSI Design
Exercise 10 - Solution

Mulitanh Linearized Differential Pair (Version 2)

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1 Introduction

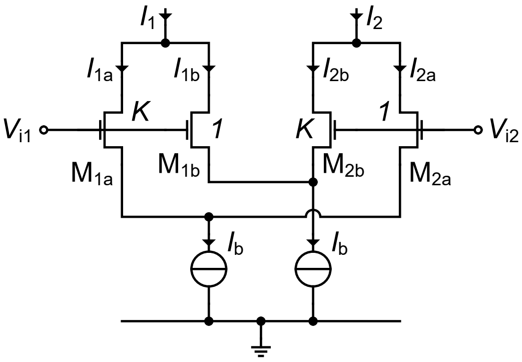

Figure 1.1: Linearized differential pair in weak inversion [1] [2].

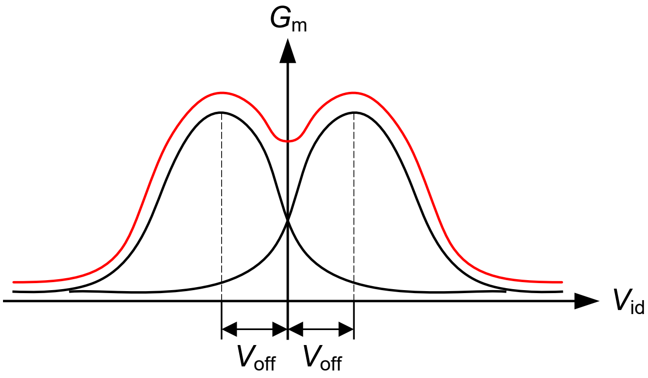

Figure 1.2: Principle of the linearized differential pair in weak inversion [1] [2].

Figure 1.1 shows a linearized differential pair operating in weak inversion [1] [2]. It is based on the principle illustrated in Figure 1.2. By making M_{2b} K-times larger than M_{1b} and M_{1a} K-times larger than M_{2a} results in introducing some offset voltage in the differential pairs M_{1a}-M_{2a} and M_{1b}-M_{2b} shifting their I-V characteristic as shown in Figure 1.2 by an offset voltage $V_{off} = n\,U_T \cdot \ln(K)$. The differential current $I_{od} \triangleq I_1 - I_2$ shows a more linear characteristics with an extended linear range [1].

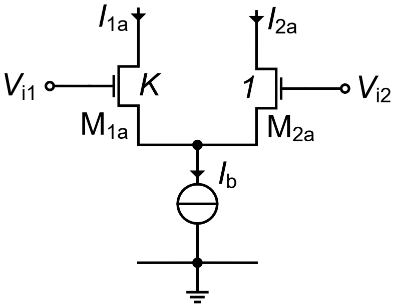

Figure 1.3: Half linearized differential pair a of Figure 1.1 in weak inversion with $\beta_{1a} = K \cdot \beta_{2a}$ [1].

The analysis of the linearized differential pair of Figure 1.1 can be done by separating the linearized differential pair into the differential pair a shown in Figure 1.3 and the differential pair b shown in Figure 1.4.

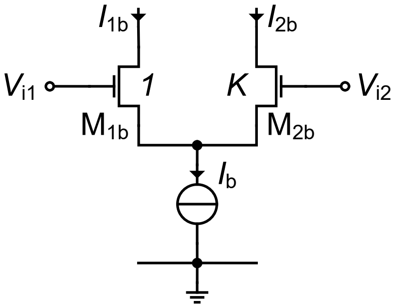

Figure 1.4: Half linearized differential pair b of Figure 1.1 in weak inversion with $\beta_{2b} = K \cdot \beta_{1b}$ [1].

2 Large-signal analysis

- Derive the expression of the differential current $I_{oda} \triangleq I_{1a} - I_{2a}$ of the half differential pair a shown in Figure 1.3 in differential mode i.e. with $V_{i1} = V_{ic} + V_{id}/2$ and $V_{i2} = V_{ic} - V_{id}/2$. Assume that both transistors are biased in weak inversion and in saturation with $\mathrm{M_{1a}}$ K-times larger than $\mathrm{M_{2a}}$ (i.e. $\beta_{1a} = K \cdot \beta_{2a}$). Simplify the analysis assuming that all slope factors are equal $n_{1a} = n_{2a} = n$. Hint: use the offset voltage $V_{off} \triangleq n\,U_T\,\ln(K)$ or $K = e^{V_{off}/(n\,U_T)}$.

Looking at Figure 1.3 we can write

$$I_{1a} = K\,I_{D0}\,e^{\frac{V_{i1}}{n\,U_T}}\,e^{-\frac{V_{Sa}}{U_T}}, \tag{2.1}$$

$$I_{2a} = I_{D0}\,e^{\frac{V_{i2}}{n\,U_T}}\,e^{-\frac{V_{Sa}}{U_T}}, \tag{2.2}$$

$$I_{1a} + I_{2a} = I_b. \tag{2.3}$$

Replacing (2.1) and (2.2) into (2.3), we get

$$e^{-\frac{V_{Sa}}{U_T}} = \frac{I_b}{I_{D0}}\,\frac{1}{K\,e^{\frac{V_{i1}}{n\,U_T}} + e^{\frac{V_{i2}}{n\,U_T}}} \tag{2.4}$$

Replacing (2.4) in (2.1) and (2.2), we get

$$I_{1a} = I_b\,\frac{K\,e^{\frac{V_{i1}}{n\,U_T}}}{K\,e^{\frac{V_{i1}}{n\,U_T}} + e^{\frac{V_{i2}}{n\,U_T}}}, \tag{2.5}$$

$$I_{2a} = I_b\,\frac{e^{\frac{V_{i2}}{n\,U_T}}}{K\,e^{\frac{V_{i1}}{n\,U_T}} + e^{\frac{V_{i2}}{n\,U_T}}}. \tag{2.6}$$

In differential mode we have

$$V_{i1} = V_{ic} + \frac{V_{id}}{2}, \tag{2.7}$$

$$V_{i2} = V_{ic} - \frac{V_{id}}{2}. \tag{2.8}$$

Replacing in (2.5) and (2.6) results in

$$I_{1a} = I_b\,\frac{K\,e^{\frac{V_{id}}{2n\,U_T}}}{K\,e^{\frac{V_{id}}{2n\,U_T}} + e^{-\frac{V_{id}}{2n\,U_T}}} = I_b\,\frac{K\,e^{\frac{V_{id}}{n\,U_T}}}{K\,e^{\frac{V_{id}}{n\,U_T}} + 1}, \tag{2.9}$$

$$I_{2a} = I_b\,\frac{e^{-\frac{V_{id}}{2n\,U_T}}}{K\,e^{\frac{V_{id}}{2n\,U_T}} + e^{-\frac{V_{id}}{2n\,U_T}}} = \frac{I_b}{K\,e^{\frac{V_{id}}{n\,U_T}} + 1}. \tag{2.10}$$

Defining

$$V_{off} \triangleq n\,U_T\,\ln(K) \tag{2.11}$$

or

$$K = e^{\frac{V_{off}}{n\,U_T}} \tag{2.12}$$

we get

$$I_{1a} = I_b\,\frac{e^{\frac{V_{id}+V_{off}}{n\,U_T}}}{e^{\frac{V_{id}+V_{off}}{n\,U_T}}+1} \tag{2.13}$$

$$I_{2a} = \frac{I_b}{e^{\frac{V_{id}+V_{off}}{n\,U_T}}+1}. \tag{2.14}$$

The differential current is then given by

$$I_{oda} \triangleq I_{1a} - I_{2a} = I_b\,\frac{e^{\frac{V_{id}+V_{off}}{n\,U_T}}-1}{e^{\frac{V_{id}+V_{off}}{n\,U_T}}+1} = I_b\,\frac{e^{\frac{V_{id}+V_{off}}{2n\,U_T}}-e^{-\frac{V_{id}+V_{off}}{2n\,U_T}}}{e^{\frac{V_{id}+V_{off}}{2n\,U_T}}+e^{-\frac{V_{id}+V_{off}}{2n\,U_T}}} = I_b\,\tanh\left(\frac{V_{id}+V_{off}}{2n\,U_T}\right) \tag{2.15}$$

Similarly for the half differential pair b shown in Figure 1.4, the currents I_{1b} and I_{2b} are given by

$$I_{1b} = I_{D0}\,e^{\frac{V_{i1}}{n\,U_T}}\,e^{-\frac{V_{Sb}}{U_T}}, \tag{2.16}$$

$$I_{2b} = K\,I_{D0}\,e^{\frac{V_{i2}}{n\,U_T}}\,e^{-\frac{V_{Sb}}{U_T}}, \tag{2.17}$$

$$I_{1b} + I_{2b} = I_b. \tag{2.18}$$

Following the same process, the differential current for differential pair b is given by

$$I_{odb} \triangleq I_{1b} - I_{2b} = I_b\,\frac{e^{\frac{V_{id}}{n\,U_T}}-e^{\frac{V_{off}}{n\,U_T}}}{e^{\frac{V_{id}}{n\,U_T}}+e^{\frac{V_{off}}{n\,U_T}}} = I_b\,\tanh\left(\frac{V_{id}-V_{off}}{2n\,U_T}\right) \tag{2.19}$$

- Using the above result, deduce the large-signal differential current $I_{od} \triangleq I_1 - I_2$ of the linearized differential pair of Figure 1.1 in differential mode.

The differential current $I_{od} \triangleq I_1 - I_2$ of the linearized differential pair of Figure 1.1 in differential mode is then given by

$$I_{od} \triangleq I_1 - I_2 = I_{1a} + I_{1b} - I_{2a} - I_{2b} = I_{1a} - I_{2a} + I_{1b} - I_{2b} = I_{oda} + I_{odb}, \tag{2.20}$$

which can be written in terms of V_{id} as

$$I_{od} = I_b\left[\tanh\left(\frac{V_{id}+V_{off}}{2n\,U_T}\right) + \tanh\left(\frac{V_{id}-V_{off}}{2n\,U_T}\right)\right]. \tag{2.21}$$

In normalized form we can write

$$i_{od} \triangleq \frac{I_{od}}{2I_b} = \frac{i_{oda}+i_{odb}}{2}, \tag{2.22}$$

where

$$i_{oda} \triangleq \frac{I_{oda}}{I_b} = \tanh\left(\frac{v_{id}+v_{off}}{2}\right), \tag{2.23}$$

$$i_{odb} \triangleq \frac{I_{odb}}{I_b} = \tanh\left(\frac{v_{id}-v_{off}}{2}\right), \tag{2.24}$$

with

$$v_{id} \triangleq \frac{V_{id}}{n\,U_T}, \tag{2.25}$$

$$v_{off} \triangleq \frac{V_{off}}{n\,U_T} = \ln(K). \tag{2.26}$$

Eqn (2.22) is plotted versus v_{id} in Figure 2.1 for different values of K. Note that $K=1$ corresponds to the normal differential pair.

The differential transconductance of the linearized differential pair is then given by

$$g_m \triangleq \frac{G_m}{G_{m0}} = \frac{g_{ma}+g_{mb}}{2} \tag{2.27}$$

where

$$g_{ma} = 1-\tanh^2\left(\frac{v_{id}+v_{off}}{2}\right), \tag{2.28}$$

$$g_{mb} = 1-\tanh^2\left(\frac{v_{id}-v_{off}}{2}\right), \tag{2.29}$$

$$G_{m0} \triangleq \frac{I_b}{2n\,U_T}. \tag{2.30}$$

Eqn (2.27) normalized to its maximum value is plotted versus v_{id} in Figure 2.2 for different values of K. Note that $K=1$ corresponds to the normal differential pair. We see that the linear range is extended. The case $K=3.73$ actually corresponds to the maximally flat case where there are no ripples in the g_m versus v_{id} characteristic. Above this value, a small ripple starts to appear.

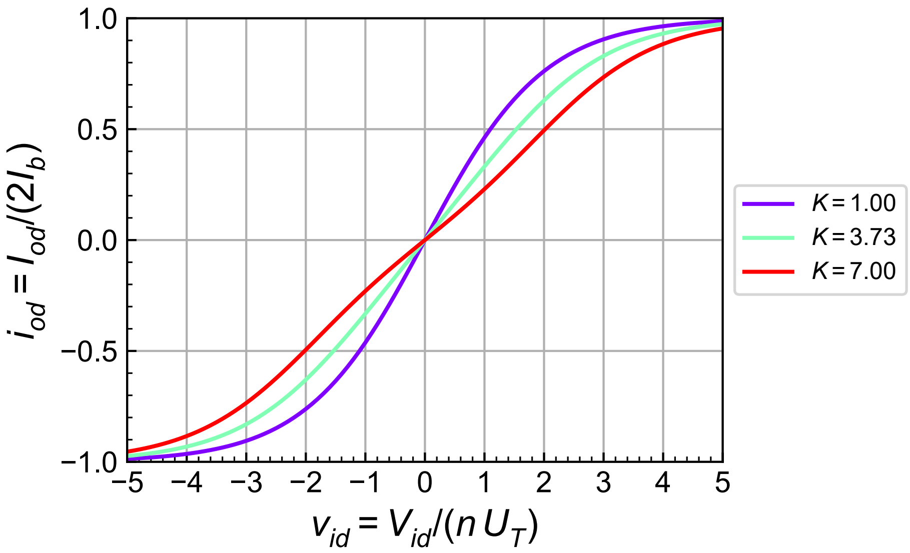

Figure 2.1: $i_{od}=I_{od}/(2I_b)$ versus $v_{id}=V_{id}/(n\,U_T)$.

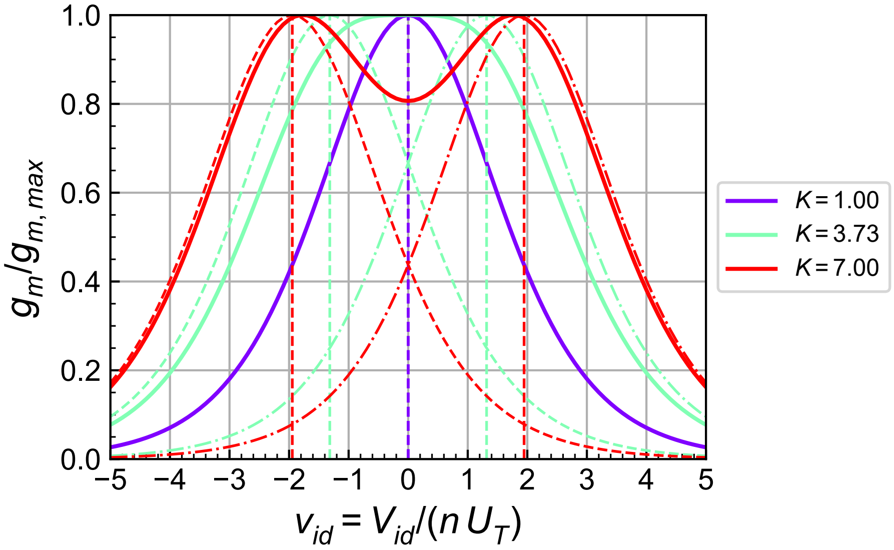

Figure 2.2: $g_m/g_{m,max}$ versus $v_{id} = V_{id}/(n\,U_T)$.

3 Small-signal analysis

- Draw the small-signal schematic of the half differential pair a shown in Figure 1.3, assuming that all the transistors are biased in weak inversion and in saturation. Derive the equivalent transconductance in differential mode $\Delta V_{i1} = -\Delta V_{i2} = \Delta V_{id}/2$

  $$G_{meqa} \triangleq \frac{\Delta I_{oda}}{\Delta V_{id}} \tag{3.1}$$

  with $\Delta I_{oda} = \Delta I_{1a} - \Delta I_{2a}$ and $\Delta V_{id} = \Delta V_{i1} - \Delta V_{i2}$. Assume that the output conductances can be neglected.

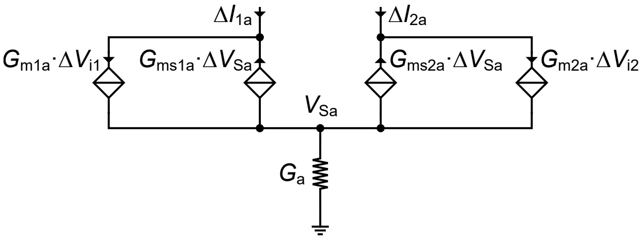

Figure 3.1: Small-signal circuit of the half-circuit shown in Figure 1.3

The small-signal circuit of the half differential pair a of Figure 1.3 is shown in Figure 3.1 where we have neglected the transistor output conductances. Contrary to the normal differential pair, for Figure 1.3 in differential mode, we can no more assume that the common source node is a virtual ground because of the assymmetry between $\mathrm{M_{1a}}$ and $\mathrm{M_{2a}}$. Assuming additionally that $G_a \ll G_{m2a} < G_{ms2a}$, the differential pair transconductance is then given by

$$G_{meq} \triangleq \frac{\Delta I_{oda}}{\Delta V_{id}} = \frac{G_{m1a}\, G_{ms2a} + G_{m2a}\, G_{ms1a}}{G_{ms1a} + G_{ms2a}}. \tag{3.2}$$

If we assume that all transistors are biased in weak inversion, the transconductances are given

$$G_{m1a} = \frac{I_{D1a}}{n\, U_T}, \tag{3.3}$$

$$G_{m2a} = \frac{I_{D2a}}{n\, U_T}, \tag{3.4}$$

$$G_{ms1a} = \frac{I_{D1a}}{U_T}, \tag{3.5}$$

$$G_{ms2a} = \frac{I_{D2a}}{U_T}. \tag{3.6}$$

$$\tag{3.7}$$

The quiescent currents I_{D1a} and I_{D2a} are given by setting $V_{id} = 0$ in (2.9) and (2.10), resulting in

$$I_{D1a} = \frac{K}{K+1}\, I_b, \tag{3.8}$$

$$I_{D2a} = \frac{I_b}{K+1}. \tag{3.9}$$

The transconductances G_{m1a} and G_{ms1a} are then related to G_{m2a} and G_{ms2a} according to

$$G_{m1a} = K\,G_{m2a}, \tag{3.10}$$

$$G_{ms1a} = K\,G_{ms2a}. \tag{3.11}$$

Replacing in (3.2) results in

$$G_{meqa} = \frac{2}{K+1}\,G_{m1a} = \frac{2K}{K+1}\,G_{m2a}. \tag{3.12}$$

We see that for $K = 1$, G_{meqa} reduces to $G_{m1a} = G_{m2a}$ as expected.

With

$$G_{m1a} = \frac{I_{D1a}}{n\,U_T} = \frac{K}{K+1}\,\frac{I_b}{n\,U_T}, \tag{3.13}$$

$$G_{m2a} = \frac{I_{D2a}}{n\,U_T} = \frac{1}{K+1}\,\frac{I_b}{n\,U_T}, \tag{3.14}$$

we can express G_{meqa} in terms of I_b according to

$$G_{meqa} = \frac{2K}{(K+1)^2}\,\frac{I_b}{n\,U_T}. \tag{3.15}$$

For $K = 1$, corresponding to the normal differential pair, we have as expected

$$G_{m1a} = G_{m2a} = \frac{I_b}{2n\,U_T}. \tag{3.16}$$

- Derive the equivalent transconductance in differential mode for the other half differential pair b

$$G_{meqb} \triangleq \frac{\Delta I_{odb}}{\Delta V_{id}} \tag{3.17}$$

with $\Delta I_{odb} = \Delta I_{1b} - \Delta I_{2b}$. Assume again that the output conductances can be neglected.

The equivalent transconductance of the other half differential pair b of Figure 1.4 is simply given by

$$G_{meqb} \triangleq \frac{\Delta I_{odb}}{\Delta V_{id}} = \frac{2K}{1+K} \cdot G_{m1b}. \tag{3.18}$$

Note that G_{m1b} is actually equal to G_{m2a}.

- Derive the equivalent small-signal transconductance $G_{meq} = \Delta I_{od}/\Delta V_{id}$ of the linearized differential pair of Figure 1.1, where $\Delta I_{od} \triangleq \Delta I_1 - \Delta I_2$ is the small-signal differential output current and $\Delta V_{id} = \Delta V_{i1} - \Delta V_{i2}$ is the small-signal input voltage. Reuse the expressions of the equivalent transconductances G_{meqa} and G_{meqb} of the half differential pairs a and b.

For the linearized differential pair, the small-signal output current is given by

$$\Delta I_{od} = \Delta I_1 - \Delta I_2 = \Delta I_{1a} + \Delta I_{1b} - \Delta I_{2a} - \Delta I_{2b} = \Delta I_{oda} + \Delta I_{odb} \tag{3.19}$$

The equivalent transconductance is the given by

$$G_{meq} \triangleq \frac{\Delta I_{od}}{\Delta V_{id}} = \frac{\Delta I_{oda}}{\Delta V_{id}} + \frac{\Delta I_{odb}}{\Delta V_{id}} = G_{meqa} + G_{meqb}. \tag{3.20}$$

Reusing expressions (3.12) and (3.18), we get

$$G_{meq} = \frac{2K}{K+1}\,(G_{m2a} + G_{m1b}) = \frac{4K}{K+1}\,G_{m2a} \tag{3.21}$$

since $G_{m1b} = G_{m2a}$. We can express G_{meq} in terms of I_b as

$$G_{meq} = \frac{4K}{(K+1)^2}\,\frac{I_b}{n\,U_T} = \frac{4K}{(K+1)^2}\,G_m. \tag{3.22}$$

For $K = 1$, corresponding to the normal differential pair, we get

$$G_{meq} = G_m = \frac{I_b}{n\,U_T}, \tag{3.23}$$

which corresponds to the transconductance G_m of the normal differential pair with a bias current equal to $2I_b$.

The transconductance reduction of the linearized differential pair compared to the normal differential pair is then given by

$$\xi \triangleq \frac{G_{meq}}{G_m} = \frac{4K}{(K+1)^2}. \tag{3.24}$$

This illustrates the trade-off between transconductance and linear range for a given current budget. As the linear range is extended by increasing K, the transconductance decreases according to (3.24).

4 Noise analysis

- Derive the output noise conductance G_{nouta} and G_{noutb} of the half differential pairs a of Figure 1.3 and Figure 1.4, respectively, in terms of the transistor noise conductances including the noise coming from the bottom bias source.

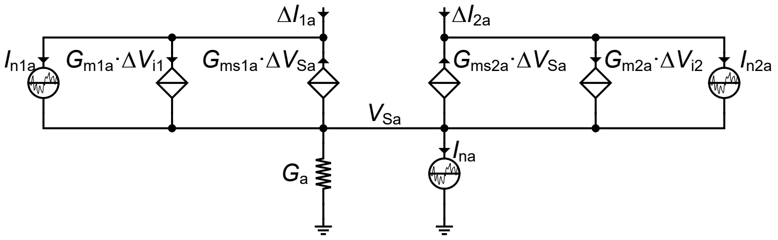

Figure 4.1: Small-signal circuit of the half-circuit shown in Figure 1.3 including noise source.

From the small-signal circuit shown in Figure 4.1, including the noise sources, we get

$$G_{nouta} = |H_{n1a}|^2 \cdot G_{n1a} + |H_{n2a}|^2 \cdot G_{n2a} + |H_{na}|^2 \cdot G_{na}, \tag{4.1}$$

where

$$G_{ni} = \gamma_{ni} \cdot G_{mi} + G_{mi}^2 \cdot \frac{\rho_n}{f\, W_i\, L_i} \quad \text{for } i = 1a, 2a, a \tag{4.2}$$

and

$$H_{n1a} = \frac{2}{K+1}, \tag{4.3}$$

$$H_{n2a} = -\frac{2K}{K+1}, \tag{4.4}$$

$$H_{na} = \frac{K-1}{K+1}. \tag{4.5}$$

which for $K = 1$ (normal differential pair) reduce to

$$H_{n1a} = 1, \tag{4.6}$$

$$H_{n2a} = -1, \tag{4.7}$$

$$H_{na} = 0. \tag{4.8}$$

We see that for $K = 1$, the noise of the differential pair directly contributes to the differential output current with a gain of 1 or -1 and the noise coming from the bottom current source is eliminated.

We get a similar result for the half differential pair b, namely

$$G_{noutb} = |H_{n1b}|^2 \cdot G_{n1b} + |H_{n2b}|^2 \cdot G_{n2b} + |H_{nb}|^2 \cdot G_{nb}, \tag{4.9}$$

where

$$G_{ni} = \gamma_{ni} \cdot G_{mi} + G_{mi}^2 \cdot \frac{\rho_n}{f\, W_i\, L_i} \quad \text{for } i = 1b, 2b, b \tag{4.10}$$

and

$$H_{n1b} = -H_{n2a} = \frac{2K}{K+1}, \tag{4.11}$$

$$H_{n2b} = -H_{n1a} = -\frac{2}{K+1}, \tag{4.12}$$

$$H_{nb} = \frac{K-1}{K+1}. \tag{4.13}$$

- Derive the output noise conductance of the linearized differential pair of Figure 1.1 using the results obtained for the half differential pair.

The output noise conductance of the linearized differential pair of Figure 1.1 is simply equal to twice that of one half differential pair

$$G_{nout} = 2G_{nouta} \tag{4.14}$$

- Calculate the input-referred noise resistance R_{nin} and the input-referred thermal noise resistance R_{nt}.

The input-referred noise resistance is obtained by dividing G_{nout} by G_{meq}^2 resulting in

$$R_{nin} = \frac{G_{nout}}{G_{meq}^2} = \frac{G_{n1a}}{2K^2\,G_{m2a}^2} + \frac{G_{n2a}}{2G_{m2a}^2} + \left(\frac{K-1}{2K}\right)^2 \frac{G_{na}}{2G_{m2a}^2}. \tag{4.15}$$

The input-referred thermal noise is obtained by replacing

$$G_{n1a} = \gamma_{n1a}\,G_{m1a} = \gamma_{n1a}\,K\,G_{m2a}, \tag{4.16}$$

$$G_{n2a} = \gamma_{n2a}\,G_{m2a}. \tag{4.17}$$

resulting in

$$R_{nt} = \frac{\gamma_{n1a}}{2K\,G_{m2a}} + \frac{\gamma_{n2a}}{2G_{m2a}} + \left(\frac{K-1}{2K}\right)^2 \frac{\gamma_{na}\,G_{ma}}{2G_{m2a}^2}. \tag{4.18}$$

- Calculate the thermal noise excess factor $\gamma_{neq} \triangleq G_{meq}\,R_{nt}$ of the linearized differential pair.

The thermal noise excess factor of the linearized differential pair is given by

$$\gamma_{neq} \triangleq G_{meq}\,R_{nt} = \frac{2\gamma_{n1a}}{K+1} + \frac{2\gamma_{n2a}\,K}{K+1} + \frac{(K-1)^2}{2K(K+1)}\,\frac{\gamma_{na}\,G_{ma}}{G_{m2a}} \tag{4.19}$$

If we assume that $\gamma_{n1a} = \gamma_{n2a} = \gamma_{na} = \gamma_n$, we get

$$\gamma_{neq} \cong 2\,\gamma_n \left[1 + \frac{(K-1)^2}{4K(K+1)}\,\frac{G_{ma}}{G_{m2a}}\right]. \tag{4.20}$$

We see that the thermal noise excess factor of the linearized differential pair is degraded by the contribution of the bias current source. Note that for $K = 3.73$, the factor $(K-1)^2/(4K(K+1))$ is equal to 0.106. This means that the contribution of the bias current source can be made negligible by making $G_{ma} \ll G_{m2a}$. In this case the thermal noise excess factor of the linearized differential pair is about equal to that of the conventional differential pair. This means that the linearization process has no impact on the thermal noise excess factor.

References

[1] H. Tanimoto, M. Koyama, and Y. Yoshida, "Realization of a 1-v active filter using a linearization technique employing plurality of emitter-coupled pairs," *IEEE Journal of Solid-State Circuits*, vol. 26, no. 7, pp. 937–945, 1991, doi: 10.1109/4.92013.

[2] J. Hauptmann, F. Dielacher, R. Steiner, C. C. Enz, and F. Krummenacher, "A low-noise amplifier with automatic gain control and anticlipping control in CMOS technology," *IEEE Journal of Solid-State Circuits*, vol. 27, no. 7, pp. 974–981, 1992, doi: 10.1109/4.142591.