Fundamentals of Electrical Circuits and Systems

# Chapter 6: AC Circuits 3

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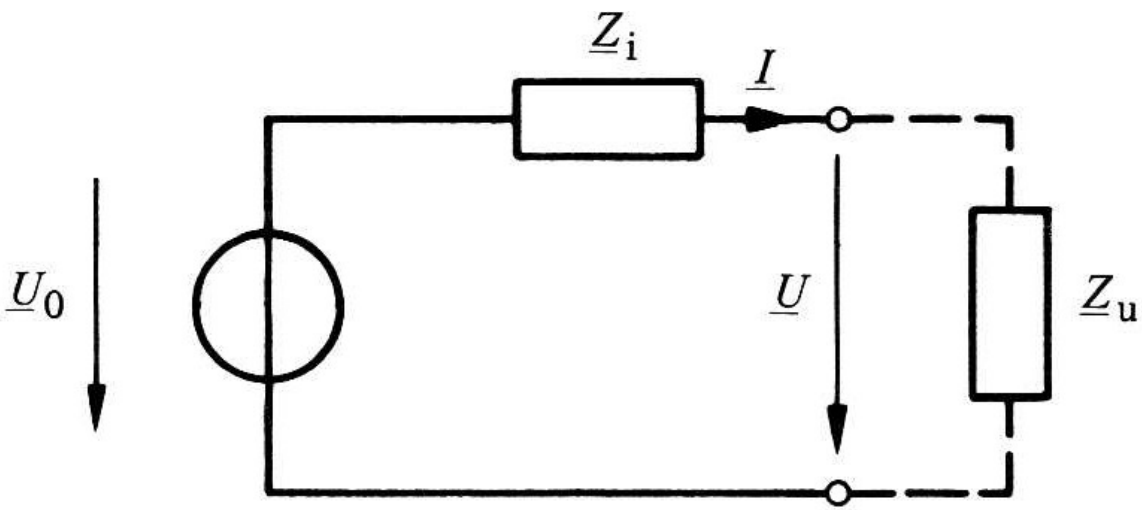
# AC Circuits 3

- Source with internal impedance
- Impedance networks
  - Impedance in series
  - Impedance in parallel
- Phasor and impedance diagram
- Voltage and current dividers
- Thévenin and Norton theorems in AC circuits
- Examples
- Instantaneous power in AC circuits
- Active power P
- Reactive power Q
- P and Q for a resistor, an inductor, a capacitor
- P and Q for an impedance
- Apparent power S
- Power factor
- Power factor correction
- Example

# Source with internal impedance

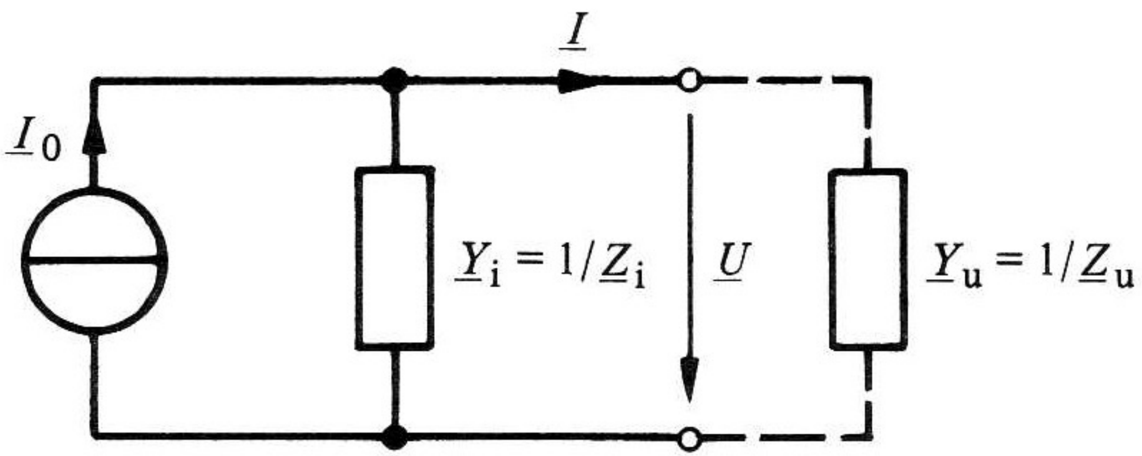
- Generalization of the notion of internal resistance: internal impedance $\underline{Z}_i = R_i + jX_i$




$$\underline{U} = \underline{U}_o - \underline{Z}_i \underline{I}$$

# Source with internal impedance

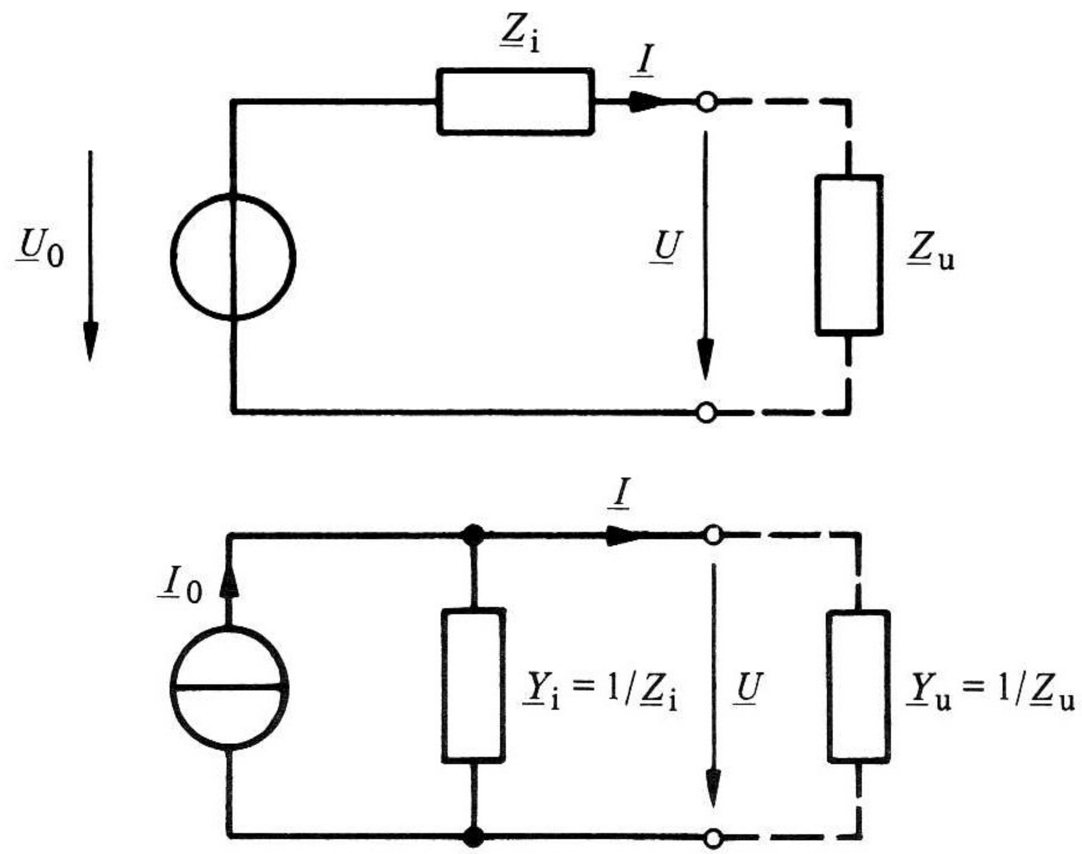
- Equivalent representation in terms of current source




$$\underline{Y}_i = 1/\underline{Z}_i$$

$$\underline{I} = \underline{I}_o - \underline{Y}_i \underline{U}$$

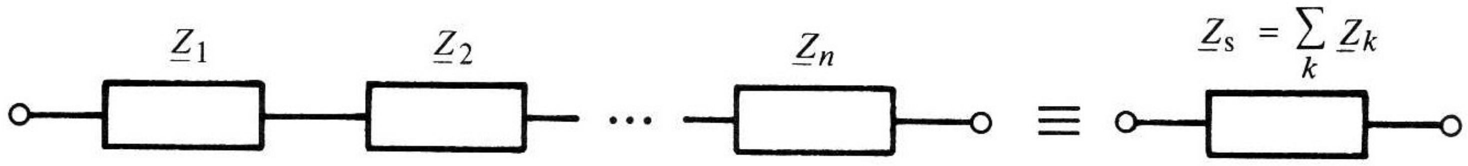
# Equivalence voltage/current sources




$$\underline{U}_o = \underline{Z}_i \underline{I}_o$$

$$\underline{I}_o = \underline{Y}_i \underline{U}_o = \frac{\underline{U}_o}{\underline{Z}_i}$$

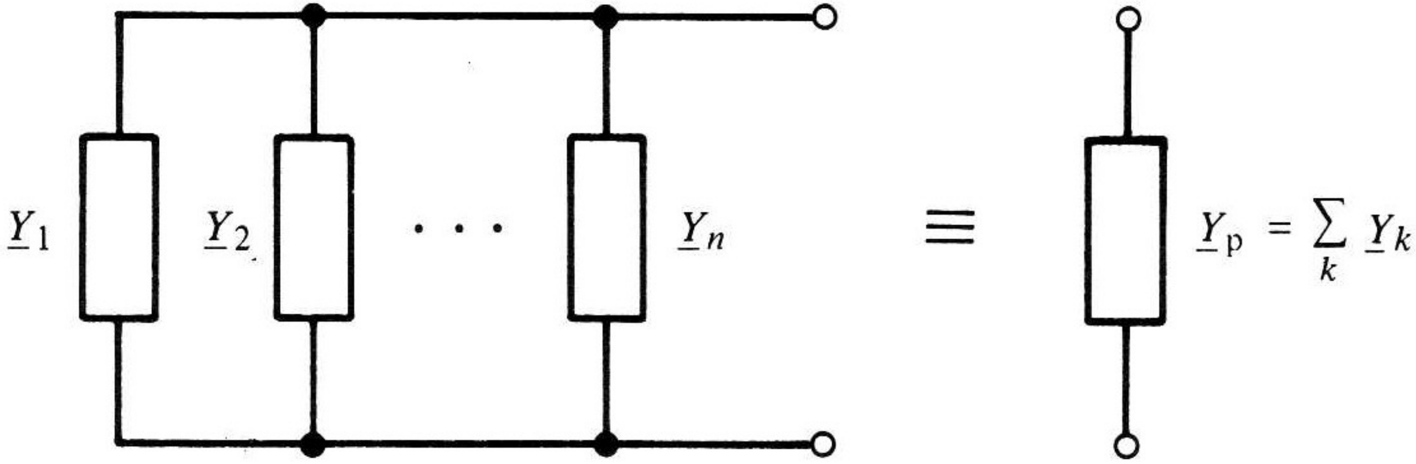
# Impedance / admittance in series




$$\underline{Z}_\mathrm{s} = \sum_{k=1}^{n} \underline{Z}_\mathrm{k}$$

$$\underline{Y}_\mathrm{s} = \frac{1}{\underline{Z}_\mathrm{s}} = \frac{1}{\sum_{k=1}^{n} \underline{Z}_\mathrm{k}} = \frac{1}{\sum_{k=1}^{n} 1/\underline{Y}_\mathrm{k}}$$

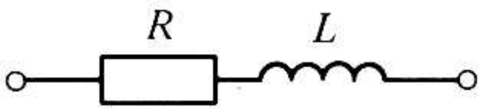
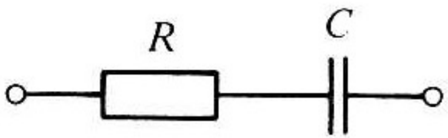
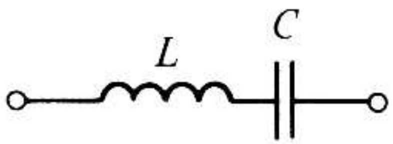
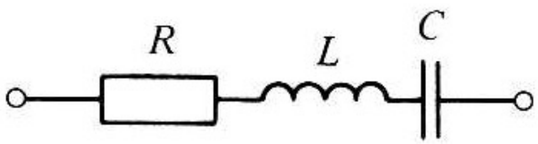
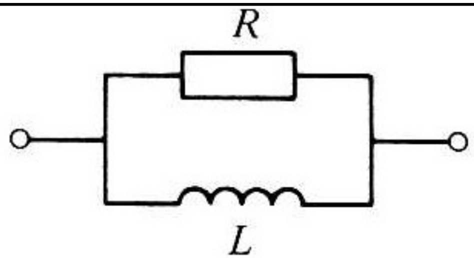
# Impedance / admittance in parallel



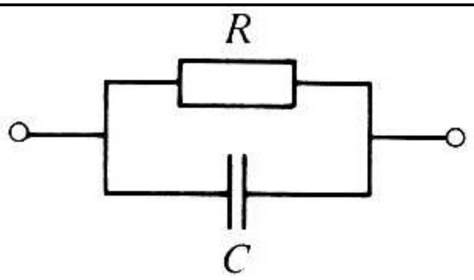
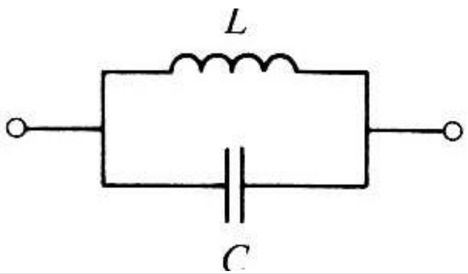
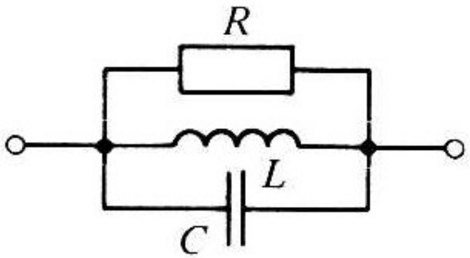

$$\underline{Y}_p = \sum_{k=1}^{n} \underline{Y}_k$$

$$\underline{Z}_p = \frac{1}{\underline{Y}_p} = \frac{1}{\sum_{k=1}^{n} \underline{Y}_k} = \frac{1}{\sum_{k=1}^{n} 1/\underline{Z}_k}$$

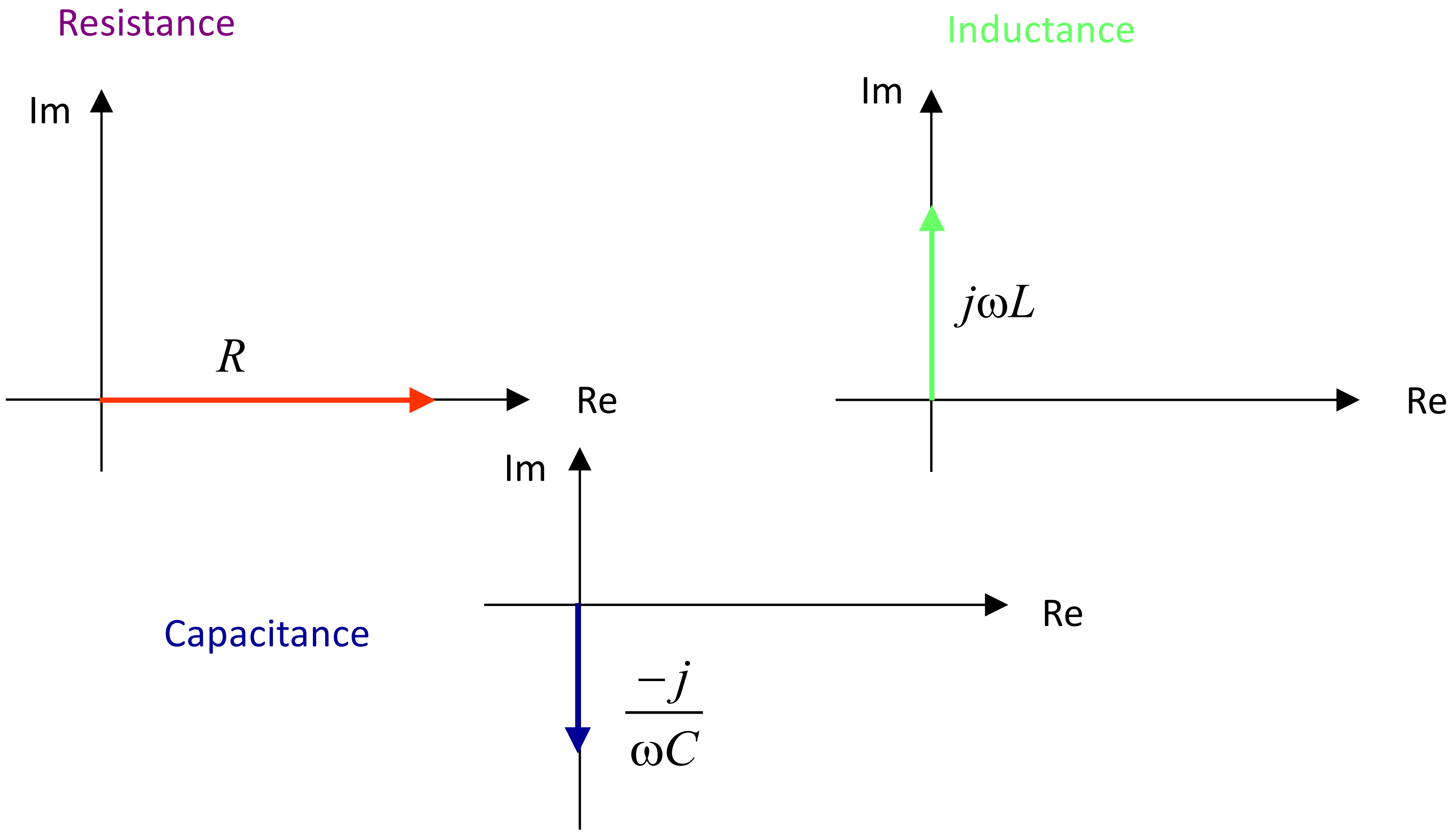
# Elementary Composite Bipoles

| Circuit | Impedance | Admittance |
|---|---|---|
|   | $R + j\omega L$ | $\dfrac{R - j\omega L}{R^2 + \omega^2 L^2}$ |
|   | $R + \dfrac{1}{j\omega C}$ | $\dfrac{R\omega^2 C^2 + j\omega C}{1 + \omega^2 R^2 C^2}$ |
|   | $j(\omega L - \dfrac{1}{\omega C})$ | $j\dfrac{\omega C}{1 - \omega^2 LC}$ |
|   | $R + j(\omega L - \dfrac{1}{\omega C})$ | $\dfrac{R - j(\omega L - 1/\omega C)}{R^2 + (\omega L - 1/\omega C)^2}$ |
|   | $\dfrac{R\omega^2 L^2 + j\omega L R^2}{R^2 + \omega^2 L^2}$ | $\dfrac{1}{R} - j\dfrac{1}{\omega L}$ |

# Elemental Composite Bipoles (continued)

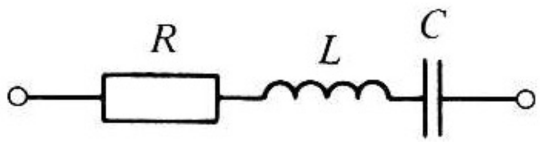
| Circuit | Impedance | Admittance |
|---|---|---|
|  | $\dfrac{R - j\omega C R^2}{1+\omega^2 R^2 C^2}$ | $\dfrac{1}{R} + j\omega C$ |
|  | $j\dfrac{\omega L}{1-\omega^2 LC}$ | $j(\omega C - \dfrac{1}{\omega L})$ |
|  | $\dfrac{R - jR^2(\omega C - 1/\omega L)}{1 + R^2(\omega C - 1/\omega L)^2}$ | $\dfrac{1}{R} - j(\omega C - \dfrac{1}{\omega L})$ |

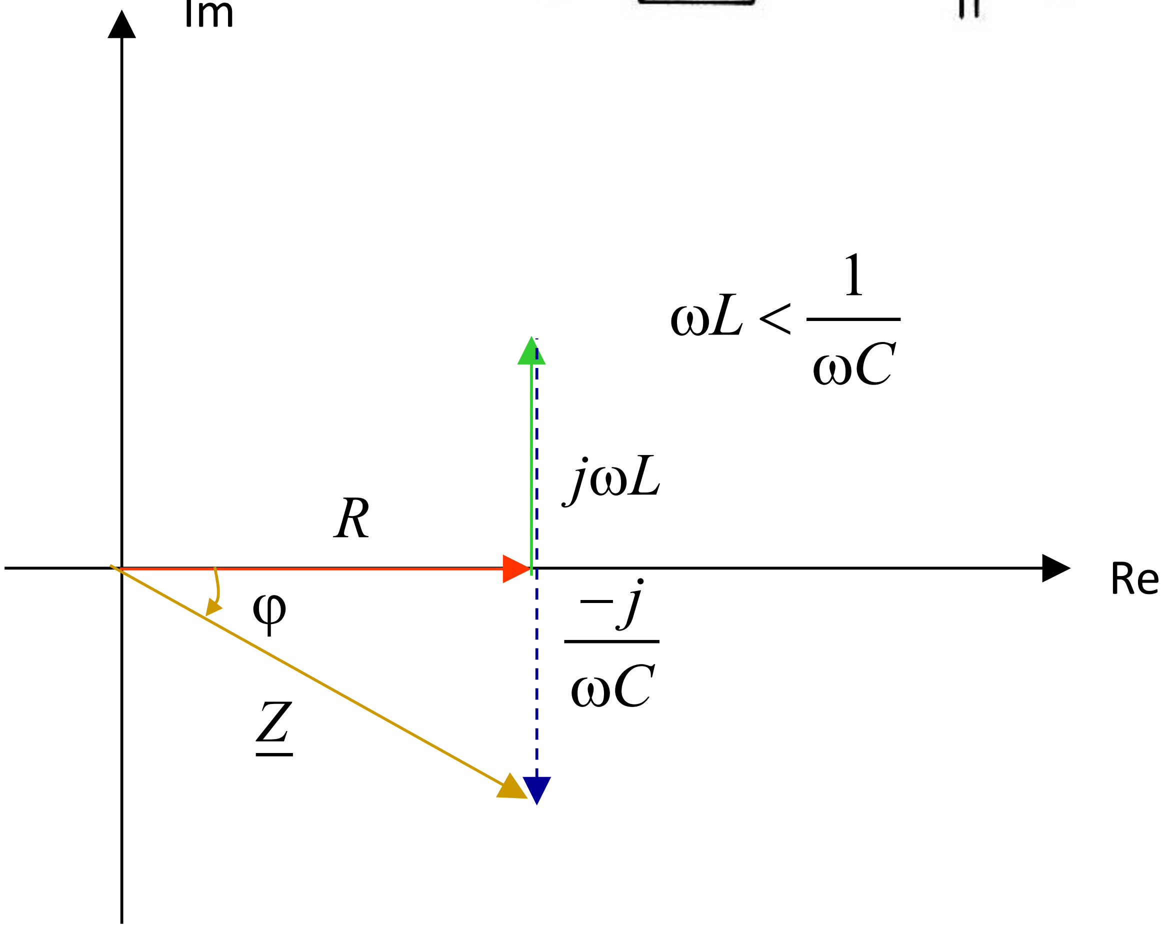
# Impedance diagram



# Impedance diagram

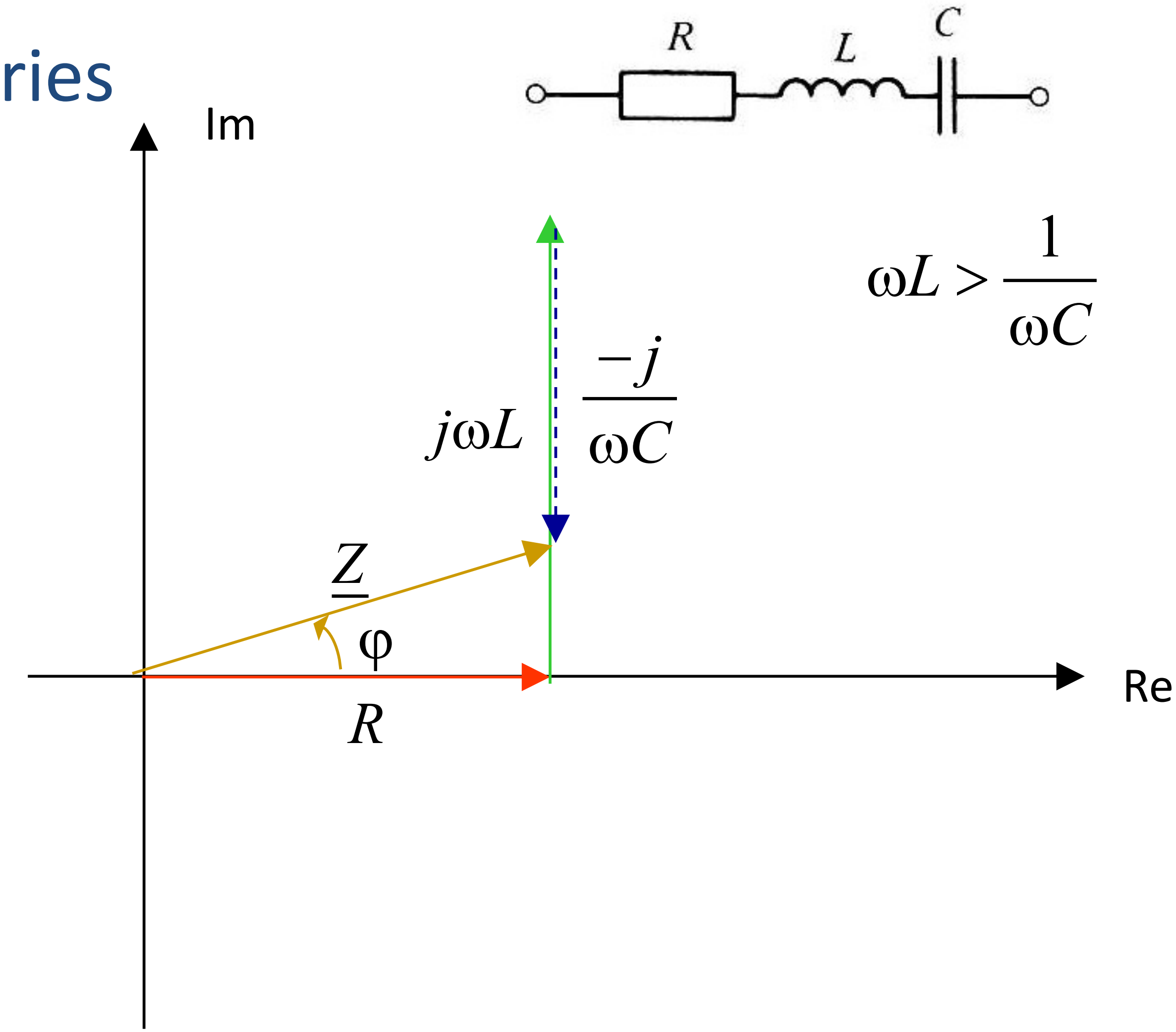
## RLC circuit in series



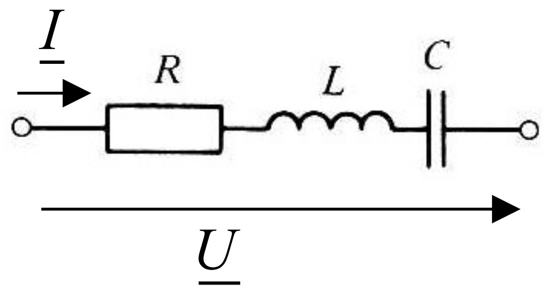
# Impedance diagram

RLC circuit in series



# Phasor diagram

## RLC circuit in series



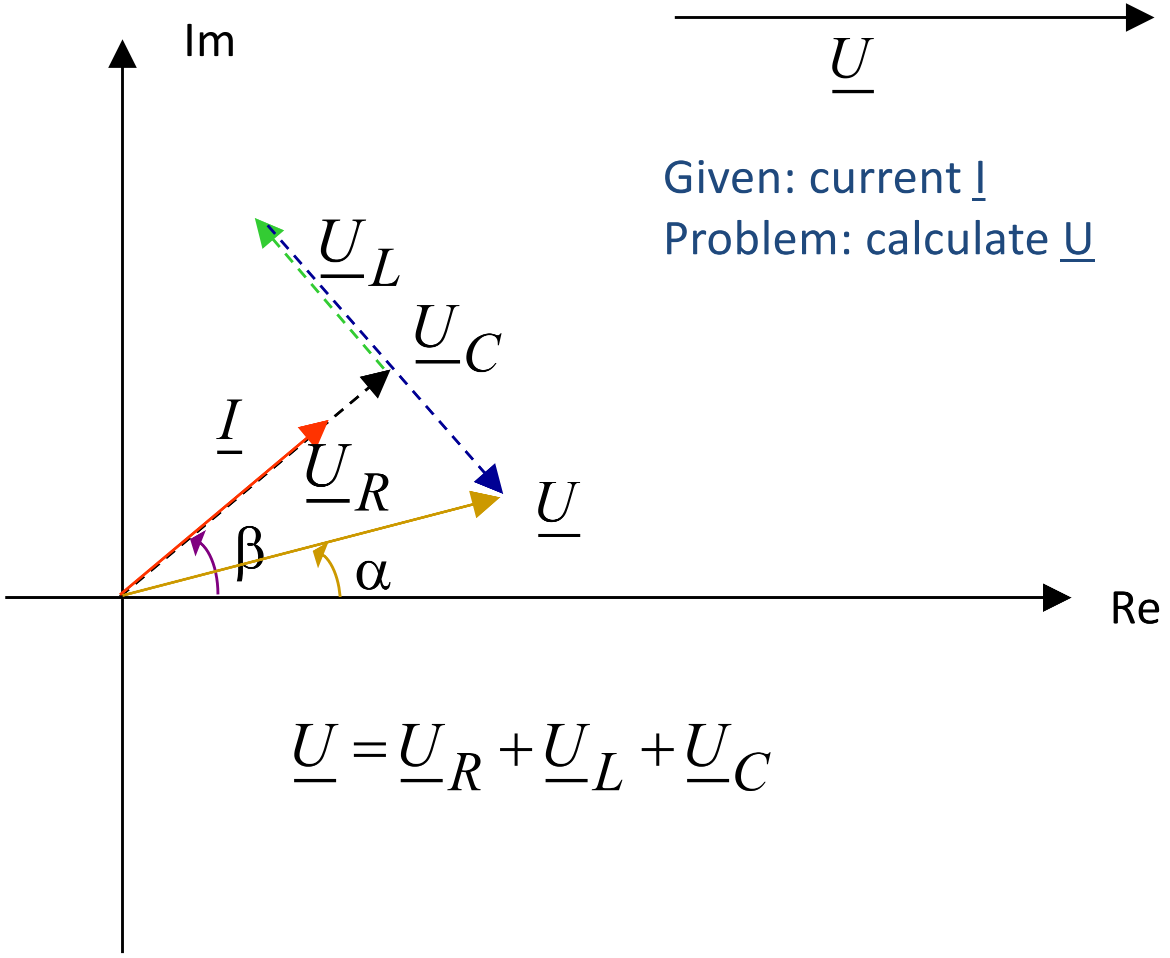

Given: current I
Problem: calculate U

$$\underline{I} = I e^{j\beta}$$

$$\underline{U}_R = R\underline{I}$$

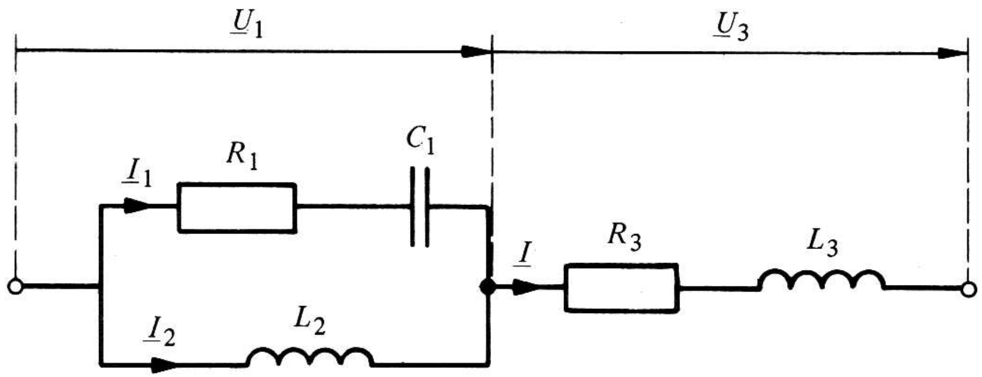
$$\underline{U}_L = j\omega L\underline{I}$$

$$\underline{U}_C = \frac{1}{j\omega C}\underline{I}$$



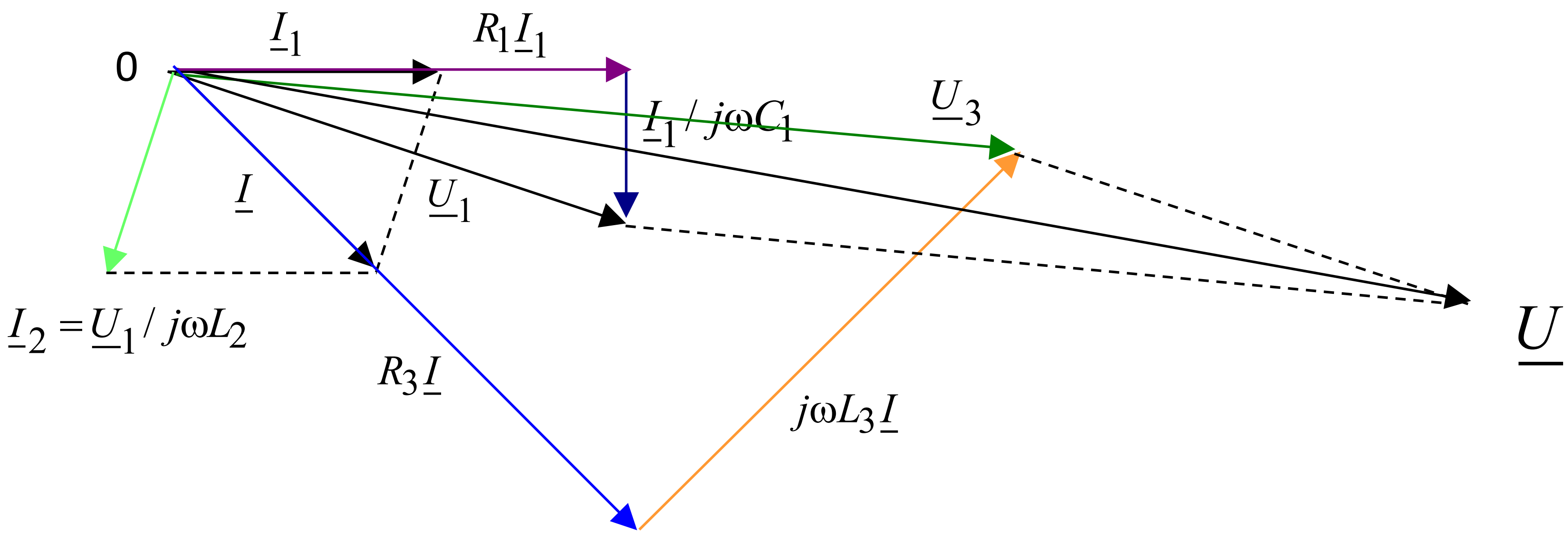

$$\underline{U} = \underline{U}_R + \underline{U}_L + \underline{U}_C$$

# Phasor diagram: series-parallel circuit




Given: current $\underline{I}_1$

Problem: calculate $\underline{U} = \underline{U}_1 + \underline{U}_3$



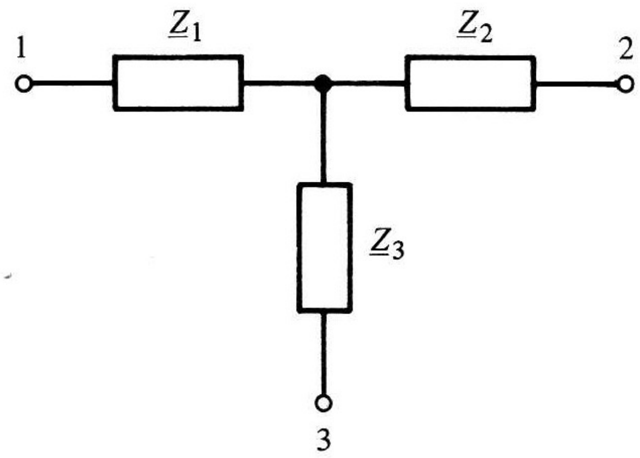
# Question

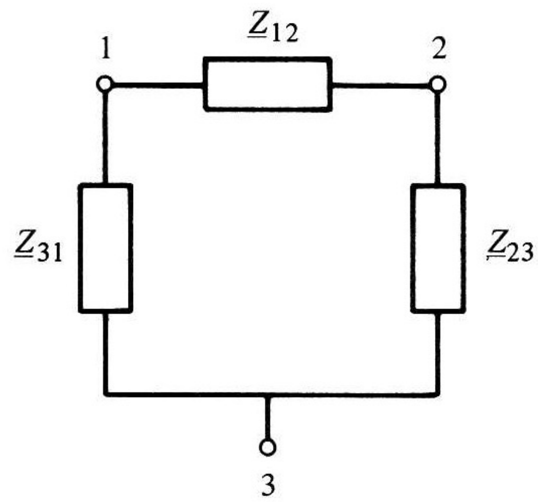
- We have two impedances in series $\underline{Z}_1$ and $\underline{Z}_2$. The module of the equivalent impedance is the sum of the modules.

A: True

B: False

# Equivalent tripoles




$$\underline{Z}_{12} = \frac{\underline{Z}_1\underline{Z}_2 + \underline{Z}_2\underline{Z}_3 + \underline{Z}_3\underline{Z}_1}{\underline{Z}_3}$$

$$\underline{Z}_{23} = \frac{\underline{Z}_1\underline{Z}_2 + \underline{Z}_2\underline{Z}_3 + \underline{Z}_3\underline{Z}_1}{\underline{Z}_1}$$

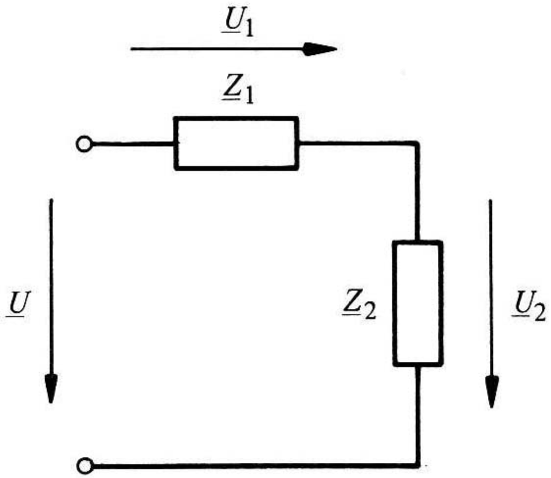
$$\underline{Z}_{31} = \frac{\underline{Z}_1\underline{Z}_2 + \underline{Z}_2\underline{Z}_3 + \underline{Z}_3\underline{Z}_1}{\underline{Z}_2}$$

$$\underline{Z}_1 = \frac{\underline{Z}_{12}\underline{Z}_{31}}{\underline{Z}_{12} + \underline{Z}_{23} + \underline{Z}_{31}}$$

$$\underline{Z}_2 = \frac{\underline{Z}_{12}\underline{Z}_{23}}{\underline{Z}_{12} + \underline{Z}_{23} + \underline{Z}_{31}}$$

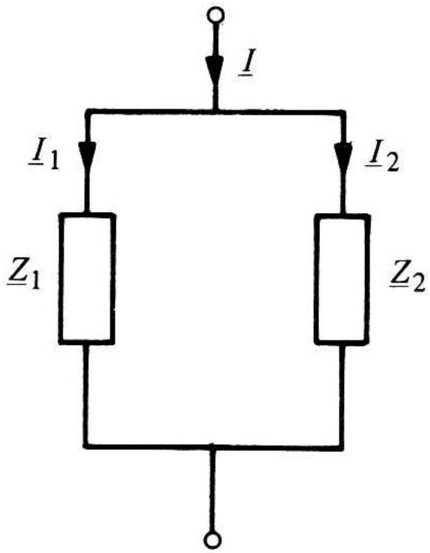
$$\underline{Z}_3 = \frac{\underline{Z}_{23}\underline{Z}_{31}}{\underline{Z}_{12} + \underline{Z}_{23} + \underline{Z}_{31}}$$

# Voltage and current dividers




$$\underline{U}_1 = \frac{\underline{Z}_1}{\underline{Z}_1 + \underline{Z}_2}\underline{U}$$

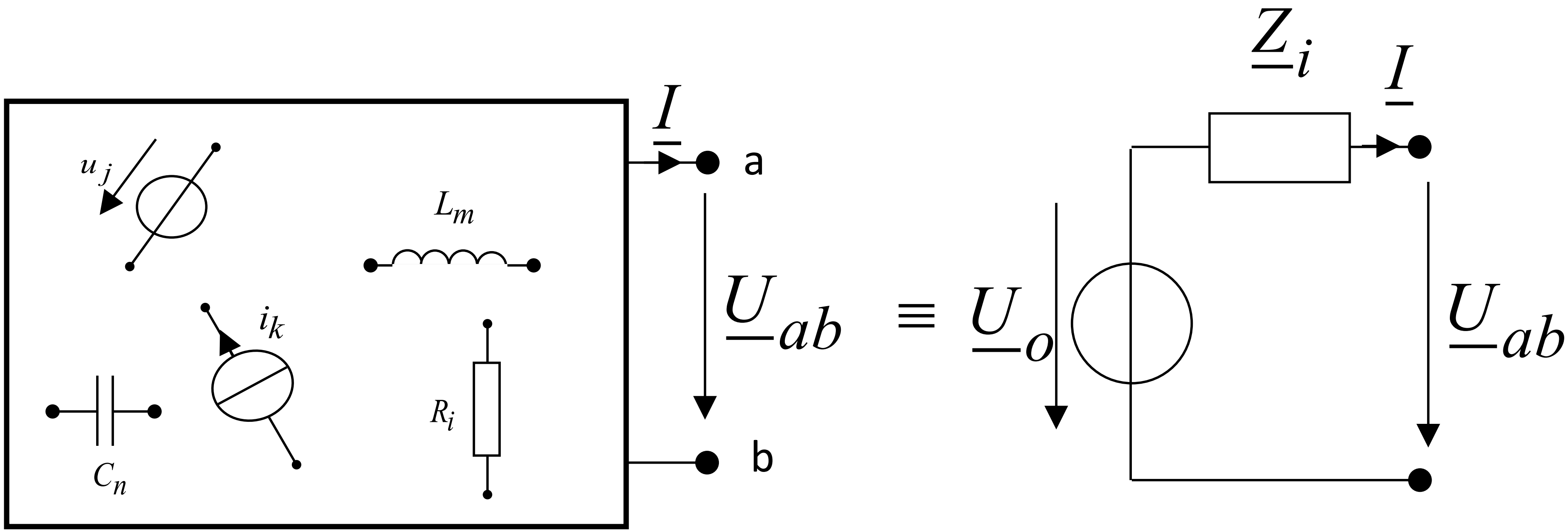
$$\underline{U}_2 = \frac{\underline{Z}_2}{\underline{Z}_1 + \underline{Z}_2}\underline{U}$$




$$\underline{I}_1 = \frac{\underline{Z}_2}{\underline{Z}_1 + \underline{Z}_2}\underline{I}$$

$$\underline{I}_2 = \frac{\underline{Z}_1}{\underline{Z}_1 + \underline{Z}_2}\underline{I}$$

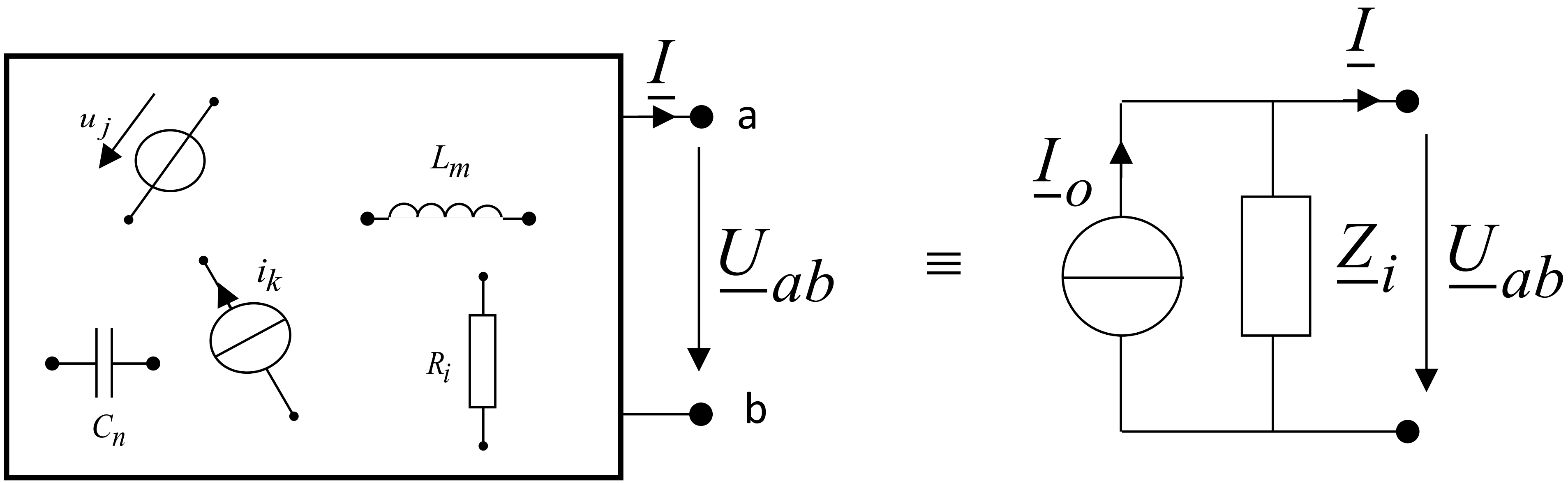
# Thévenin's theorem in AC circuits




All sources have the same frequency

$$\underline{U}_o = \underline{U}_{ab}\Big|_{\underline{I}=0} \qquad \underline{Z}_i = \underline{Z}_{ab}\Big|_{u_j=0,\, i_k=0}$$

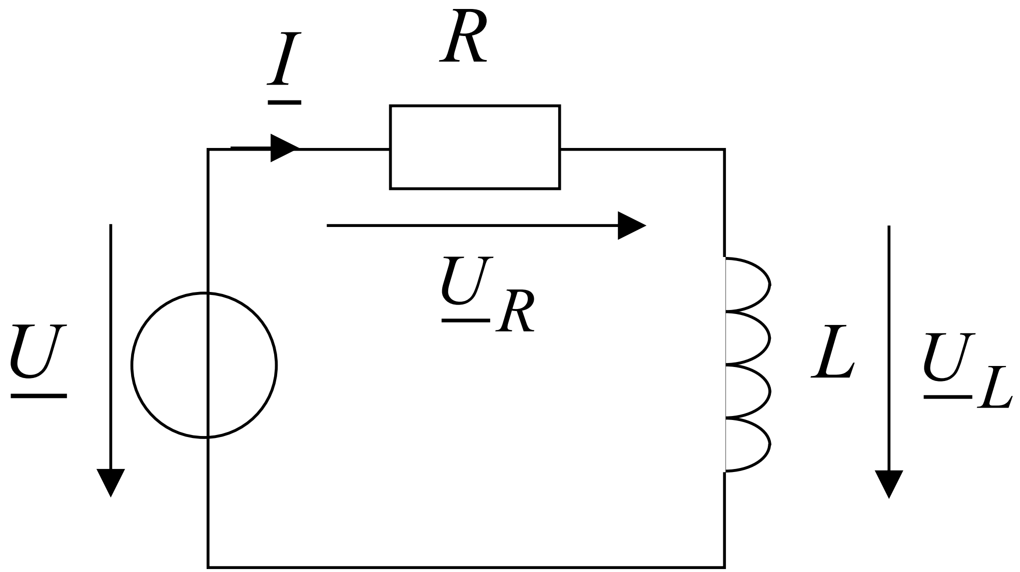
# Norton's theorem in AC circuits




All sources have the same frequency

$$\underline{I}_o = \underline{I}\Big|_{\underline{U}_{ab}=0} \qquad \underline{Z}_i = \underline{Z}_{ab}\Big|_{u_j=0,\, i_k=0}$$

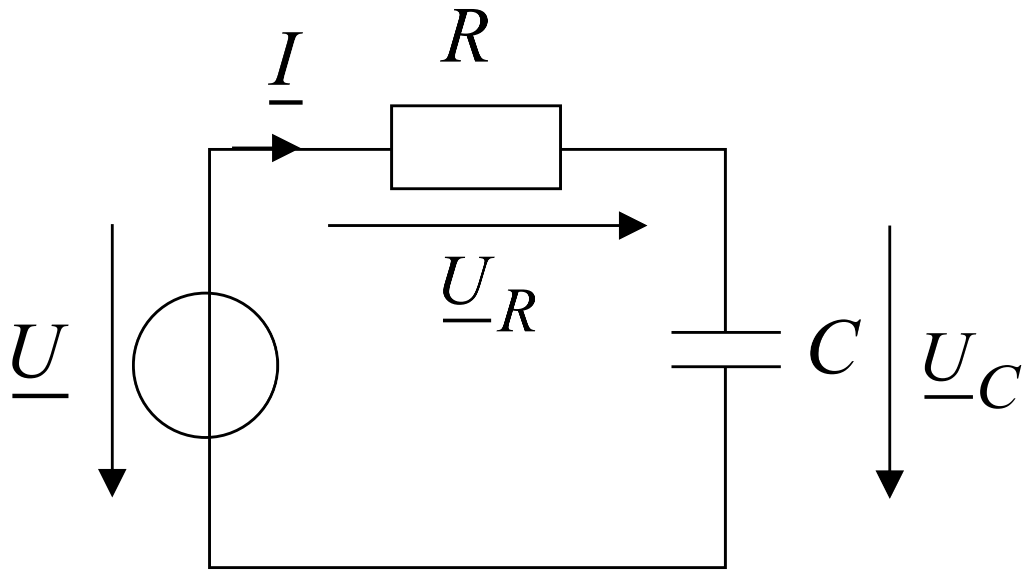
# Example 1: RL circuit in series




$$\underline{U} = Ue^{j\theta}$$

Determine $\underline{I}$, $\underline{U}_R$ and $\underline{U}_L$

# Example 2: RC circuit in series



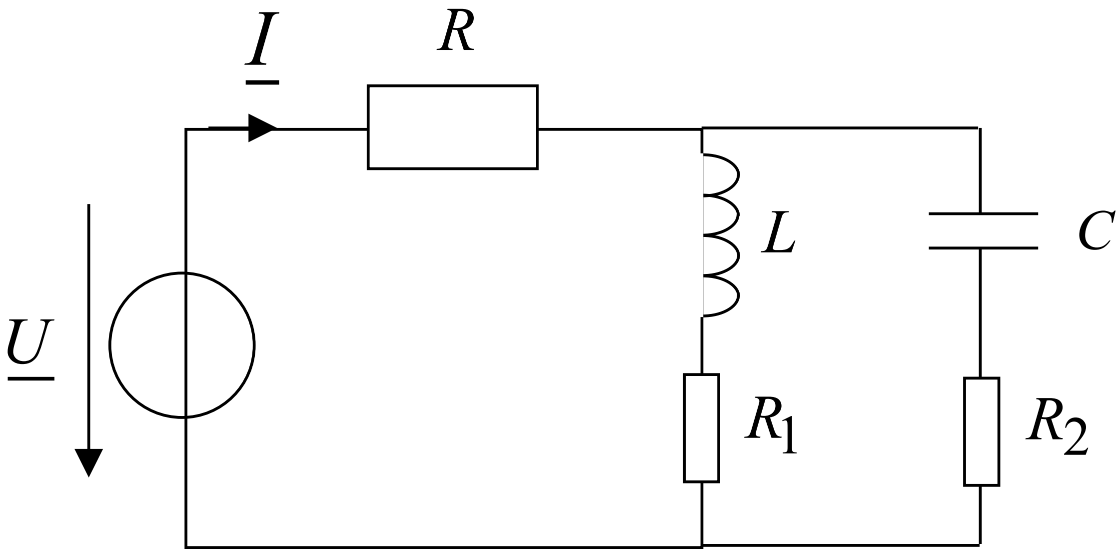

$$R = 5\,\Omega$$

$$\underline{Z}_C = -j8.66\,\Omega$$

$$\underline{U} = 20e^{j45^\circ}$$

Determine $\underline{I}$, $\underline{U}_R$ and $\underline{U}_C$

# Example 3: RLC circuit




$$\underline{U} = 10e^{j15^{\circ}}, \ \omega = 500 \text{ rad/s}$$

$$R = 2\,\Omega$$

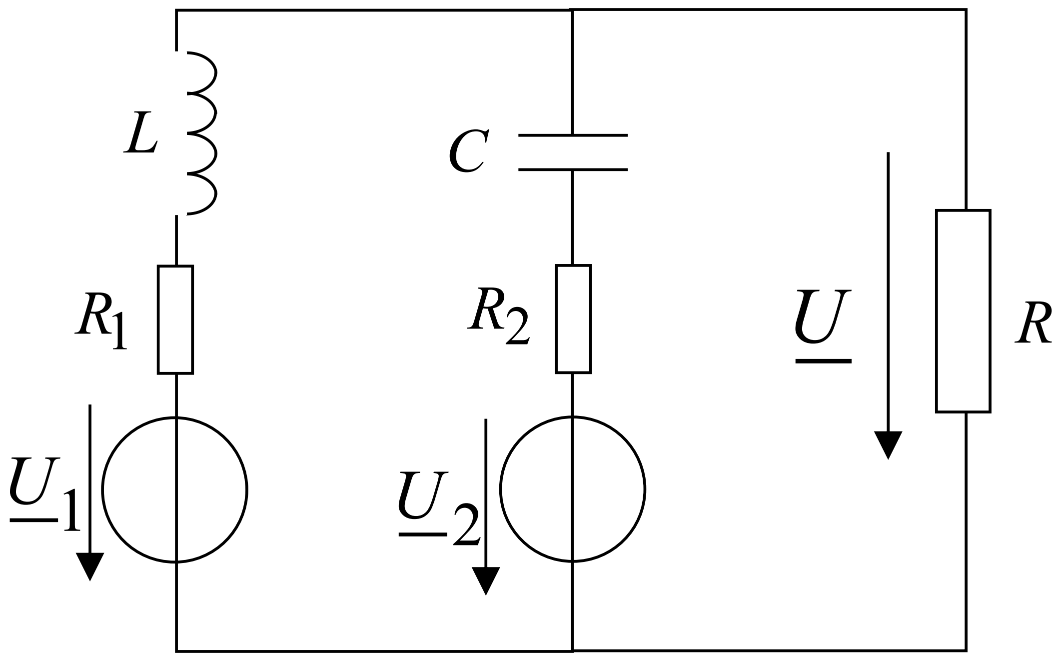
$$L = 2 \text{ mH}$$

$$C = 1000\ \mu\text{F}$$

$$R_1 = R_2 = 1\,\Omega$$

Determine the current *i*(*t*)

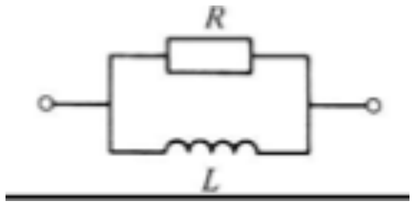
# Example 4: Thévenin's theorem




Determine the voltage $\underline{U}$

# Question

- The resistance of the following impedence is




A: $R$

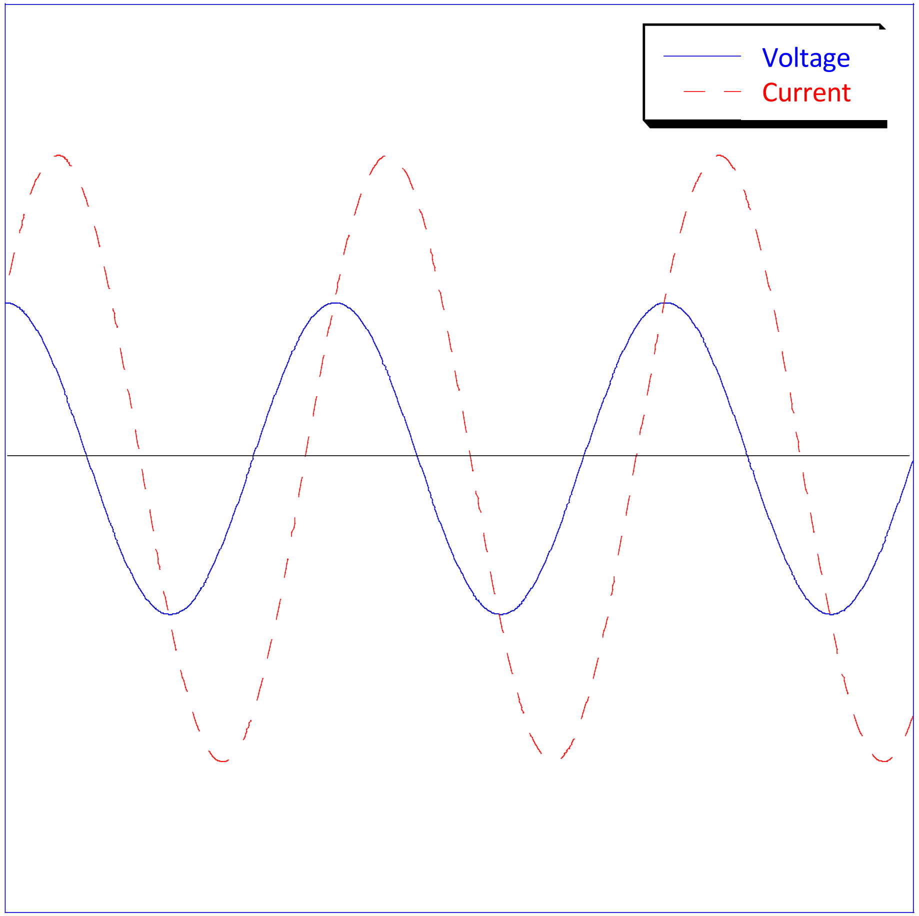
B: $\dfrac{R\omega^2 L^2}{R^2 + \omega^2 L^2}$

C: $\sqrt{R^2 + \omega^2 L^2}$

# Instantaneous power in AC circuits

$$u(t) = U\sqrt{2}\cos(\omega t + \alpha)$$

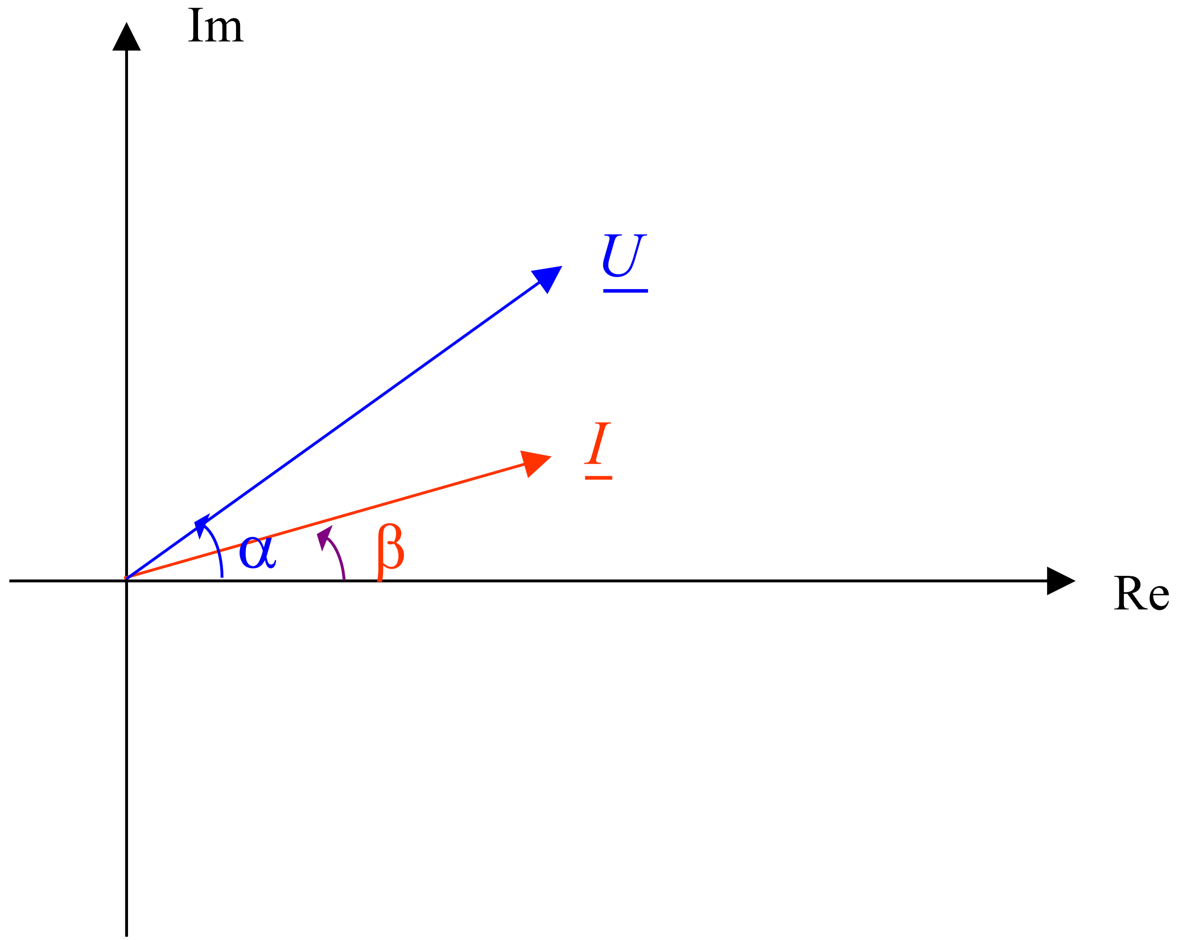
$$i(t) = I\sqrt{2}\cos(\omega t + \beta)$$



# Instantaneous power in AC circuits

$$\underline{U} = Ue^{j\alpha}$$

$$\underline{I} = Ie^{j\beta}$$



# Instantaneous power in AC circuits

$$u(t) = U\sqrt{2}\cos(\omega t + \alpha)$$

$$i(t) = I\sqrt{2}\cos(\omega t + \beta)$$

Instantaneous power:

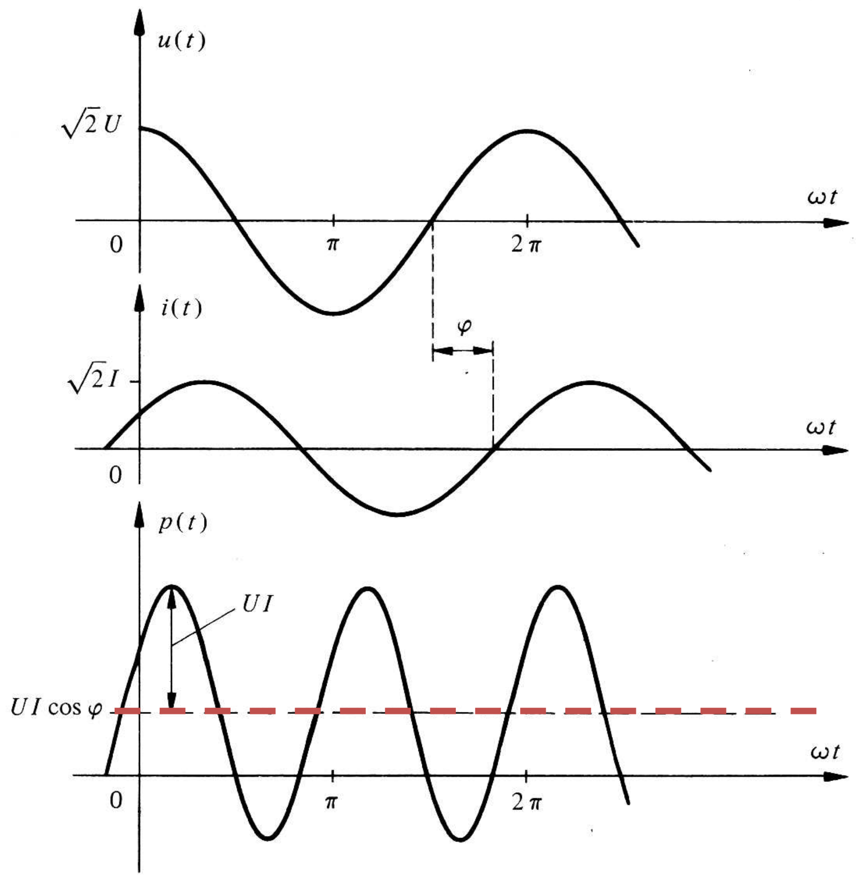
$$\begin{aligned} p(t) &= u(t)i(t) = 2UI\cos(\omega t + \alpha)\cos(\omega t + \beta) \\ &= UI[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)] \\ &= \underbrace{UI\cos\varphi}_{\text{Constant component}} + \underbrace{UI\cos(2\omega t + \alpha + \beta)}_{\text{Sinusoidal component (double frequency)}} \end{aligned}$$

# Instantaneous power in AC circuits

$$u(t) = U\sqrt{2}\cos(\omega t + \alpha)$$

$$i(t) = I\sqrt{2}\cos(\omega t + \beta)$$

$$p(t) = UI\cos\varphi + UI\cos(2\omega t + \alpha + \beta)$$



# Instantaneous power in AC circuits

Using the following trigonometric identity:

$$\cos(2\omega t + \alpha + \beta) = \cos(2\omega t + 2\alpha - \varphi)$$
$$= \cos\varphi \cos(2\omega t + 2\alpha) + \sin\varphi \sin(2\omega t + 2\alpha)$$

Instantaneous power becomes:

$$p(t) = \underbrace{UI\cos\varphi[1 + \cos(2\omega t + 2\alpha)]} + \underbrace{UI\sin\varphi\sin(2\omega t + 2\alpha)}$$

Pulsed component, positive, which oscillates around UIcosϕ

Alternative component with average value of zero

Translates a unidirectional energy exchange

Translates an oscillatory energy exchange

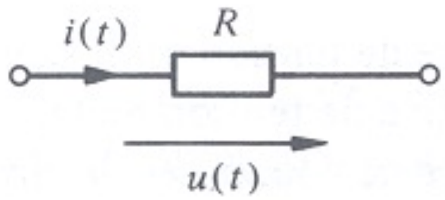
# Active and reactive power

$$p(t) = UI\cos\varphi\left[1+\cos\left(2\omega t+2\alpha\right)\right]+UI\sin\varphi\sin\left(2\omega t+2\alpha\right)$$
$$= P\left[1+\cos\left(2\omega t+2\alpha\right)\right]+Q\sin\left(2\omega t+2\alpha\right)$$

| | | |
|---|---|---|
| Active power | $P = UI\cos\varphi$ | W |
| Reactive power | $Q = UI\sin\varphi$ | var (volt-ampere-reactive) |

Reactive power : storage and periodic return of energy at twice the frequency of current and voltage

# P and Q for a resitor




$$\underline{U} = R\underline{I}$$

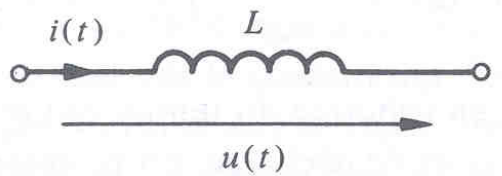
$$\underline{Z} = R, \quad \varphi = 0$$

Active power $$P = UI = \frac{U^2}{R} = RI^2$$

Reactive power $$Q = 0$$

Instantaneous power : $$p(t) = P\left(1 + \cos(2\omega t)\right)$$

# P and Q for an inductor




$$\underline{U} = j\omega L \underline{I}$$

$$\underline{Z} = j\omega L = jX_L\,, \quad \varphi = \pi/2$$

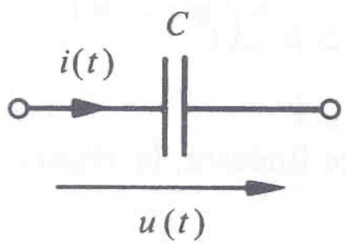
Active power $$P = 0$$

Reactive power $$Q = UI = \frac{U^2}{X_L} = X_L I^2$$

Instantaneous power : $$p(t) = Q\sin(2\omega t + 2\alpha)$$

An inductor does not dissipate energy: energy is absorbed during one half of each half cycle and returned during the other half.

# P and Q for a capacitor




$$\underline{I} = j\omega C \underline{U}$$

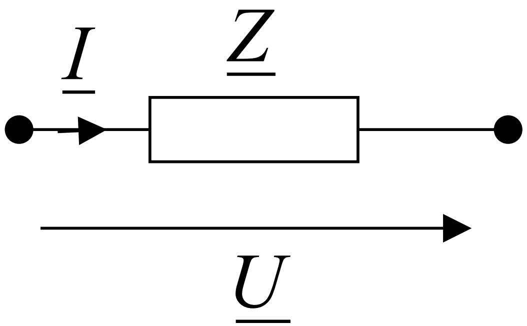
$$\underline{Z} = 1/j\omega C = jX_C, \quad \varphi = -\pi/2$$

Active power $$P = 0$$

Reactive power $$Q = -UI = \frac{U^2}{X_C} = X_C I^2$$

Instantaneous power : $$p(t) = Q\sin(2\omega t + 2\alpha)$$

# P and Q for an impedance




$$\underline{U} = \underline{Z}\,\underline{I}$$

$$\underline{Z} = Ze^{j\varphi} = R + jX$$

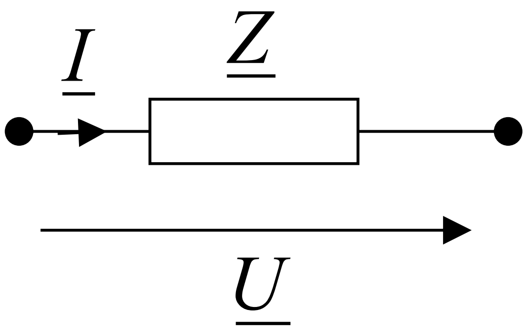
Active power
$$P = UI\cos\varphi = RI^2$$

Reactive power
$$Q = UI\sin\varphi = XI^2$$

Instantaneous power :

$$p(t) = P\left[1 + \cos\left(2\omega t + 2\alpha\right)\right] + Q\sin\left(2\omega t + 2\alpha\right)$$

# Question




$$P = UI\cos\varphi = RI^2$$

What would be the expression of the active power expressed as a function of the voltage?

A. Can not express it according to the voltage

B. $P = U^2 / R$

C. $P = RU^2 / Z^2$

D. None of the above

# Complex apparent power

The complex apparent power, noted $\underline{S}$:

- The real part of $\underline{S}$ corresponds to P
- The imaginary part of $\underline{S}$ corresponds to Q
- Used to represent P and Q in complex form

$$\underline{S} = P + jQ$$

The apparent power unit is the voltampere (VA)

# $\underline{S}$ for an impedance

$$\underline{Z} = Ze^{j\varphi} = R + jX$$

$$P = UI\cos\varphi$$

$$Q = UI\sin\varphi$$

$$\Rightarrow \quad \underline{S} = P + jQ$$

$$= UI\cos\varphi + jUI\sin\varphi$$

$$= UI(\cos\varphi + j\sin\varphi)$$

$$= UIe^{j\varphi} = Se^{j\varphi}$$

$$S = \sqrt{P^2 + Q^2}$$

# $\underline{S}$ for an impedance

Taking the voltage as reference:

$$\underline{U} = U$$

We can write :

$$\underline{I} = Ie^{-j\varphi}$$

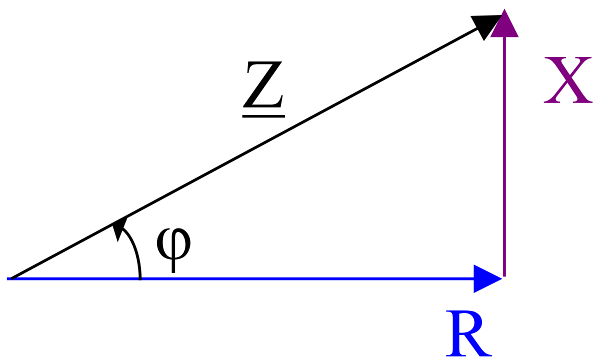
And then :

$$\underline{I}^{*} = Ie^{j\varphi}$$

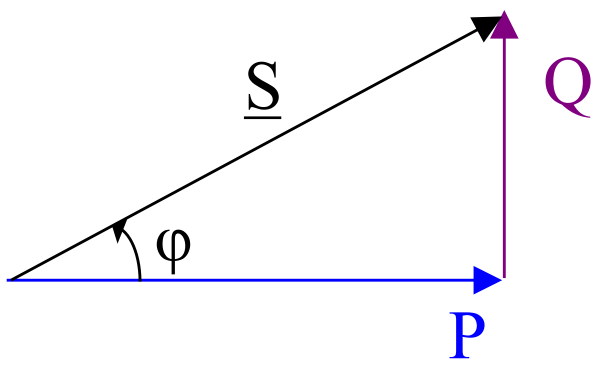
$$\Longrightarrow \quad \underline{S} = \underline{U}\,\underline{I}^{*}$$

# Apparent power

Triangle of impedances

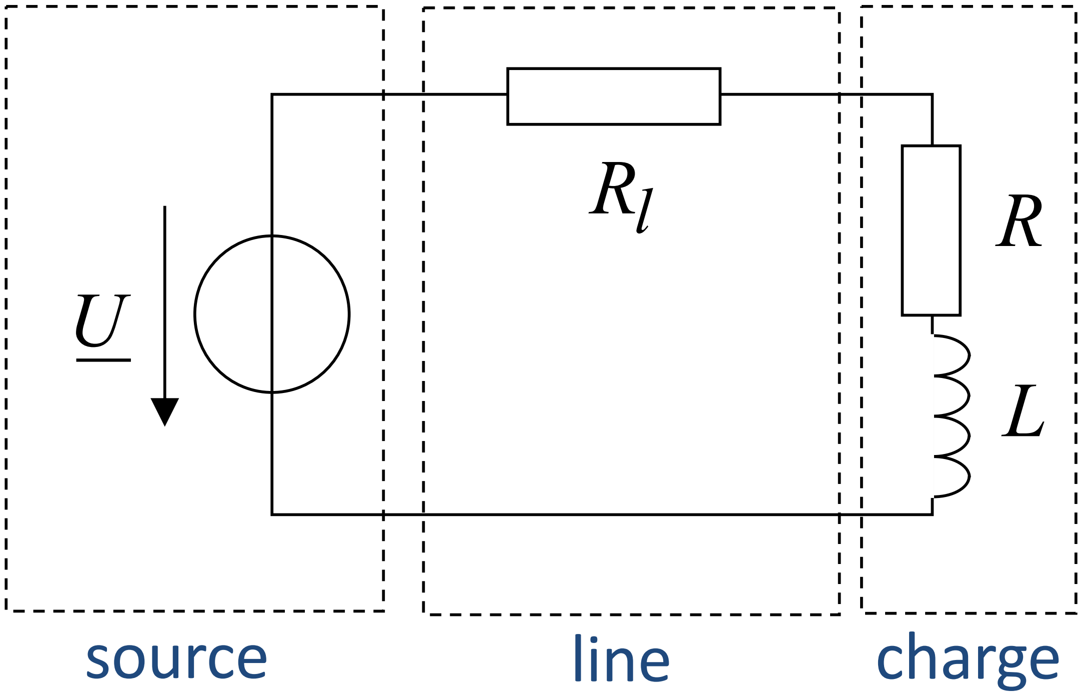



And of powers



# Reactive power: example

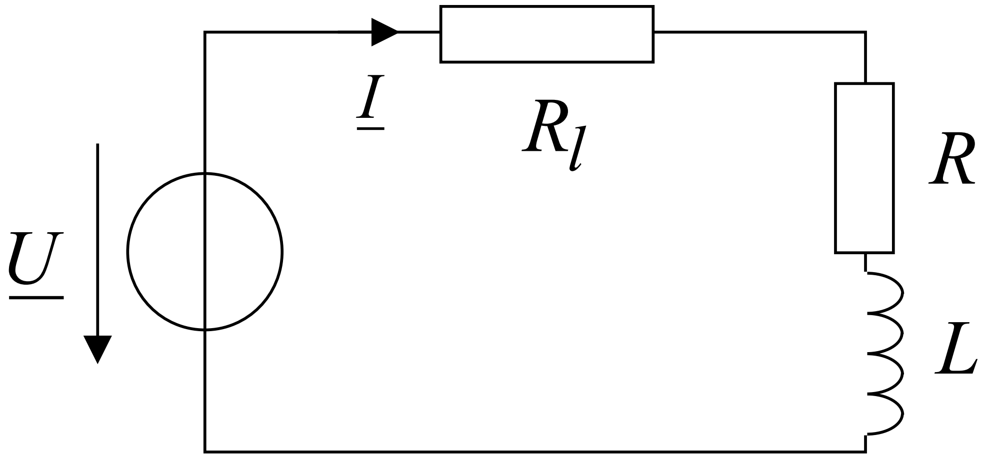
- Single-phase line, RL charge in series



# Reactive power: example (continued)

$$I = \frac{U}{\sqrt{(R_l + R)^2 + (\omega L)^2}}$$

$$\sin\varphi = \frac{\omega L}{\sqrt{(R_l + R)^2 + (\omega L)^2}}$$



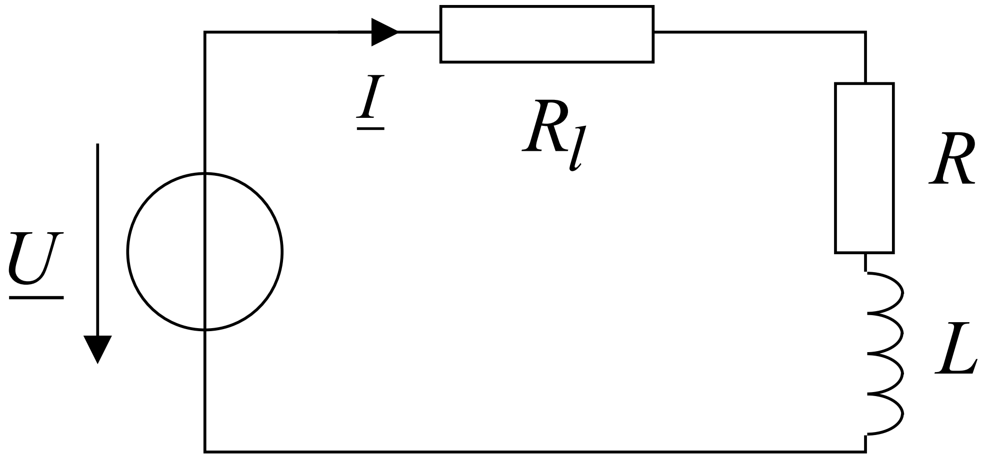

$$R_l << R \quad \Rightarrow \quad I \cong \frac{U}{\sqrt{R^2 + (\omega L)^2}}$$

$$\sin\varphi \cong \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}}$$

# Reactive power: example (continued)

$$I \cong \frac{U}{\sqrt{R^2 + (\omega L)^2}}$$

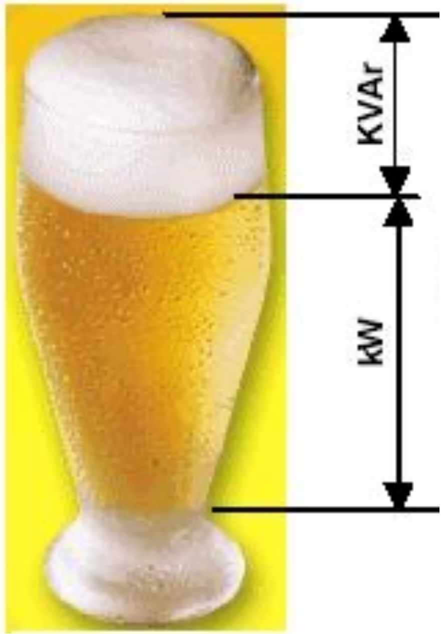
$$\sin\varphi \cong \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}}$$




Losses in the line: $P_l = R_l I^2 = R_l I \frac{U}{\sqrt{R^2 + (\omega L)^2}} = \frac{R_l U I \sin\varphi}{\omega L}$

$\Longrightarrow$ $P_l = \frac{R_l}{\omega L} Q$ $\Longrightarrow$ Losses in the line are worth, to a factor, the reactive power

# Power: analogy



# Power factor

The power factor is defined by

$$F_p = \cos\varphi$$

ϕ being between –π/2 and π/2, the power factor is always positive and between 0 and 1

Resistance: $F_p = 1$

Inductance: $F_p = 0$

Capacitance: $F_p = 0$

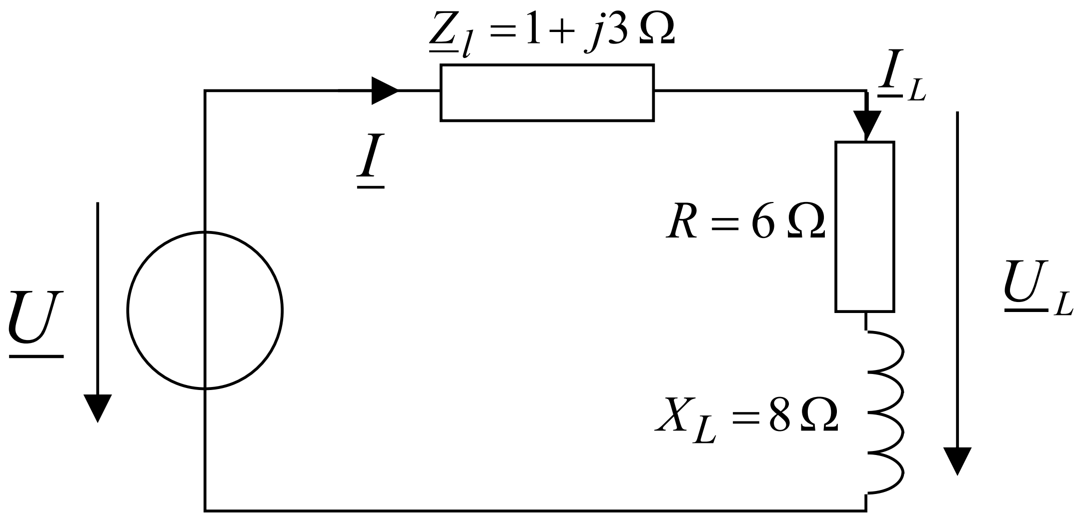
# Improving the power factor

- In general, in industry, charges are inductive in nature
- To get the most out of the equipment, the actual power must be as close as possible to the apparent power, i.e. the reactive power must be minimized

# Improving the power factor

- In the triangle of powers, the length S must tend towards the one of P and ϕ must be as small as possible.
- This angle is reduced by adding capacitors in parallel with the charge: this is the correction of the power factor.

# Active and reactive powers: example




An industrial charge is represented by an impedance formed by the series setting of a resistor and an inductor. The voltage across the charge is $\underline{U}_L$= 250e$^{j0}$ (V).

a) Calculate the current $\underline{I}_L$, Q, P, S and $F_p$ of the charge.

b) Calculate the voltage of the source if the line connecting the source to the charge has a given impedance $\underline{Z}_l$. Calculate the power lost in the line.

c) If we add a capacitor of $X_C$ =-12.5 Ω in parallel, calculate the current taken by the C, the new current supplied by the source and $F_p$ of the entire charge and capacity (for the same voltage across the charge).

d) Calculate the new voltage of the source and the new power loss in the line.

# Solution

a)

$$\underline{Z}_{ch} = 6 + j8 = 10e^{j53.1^{o}} \ \Omega$$

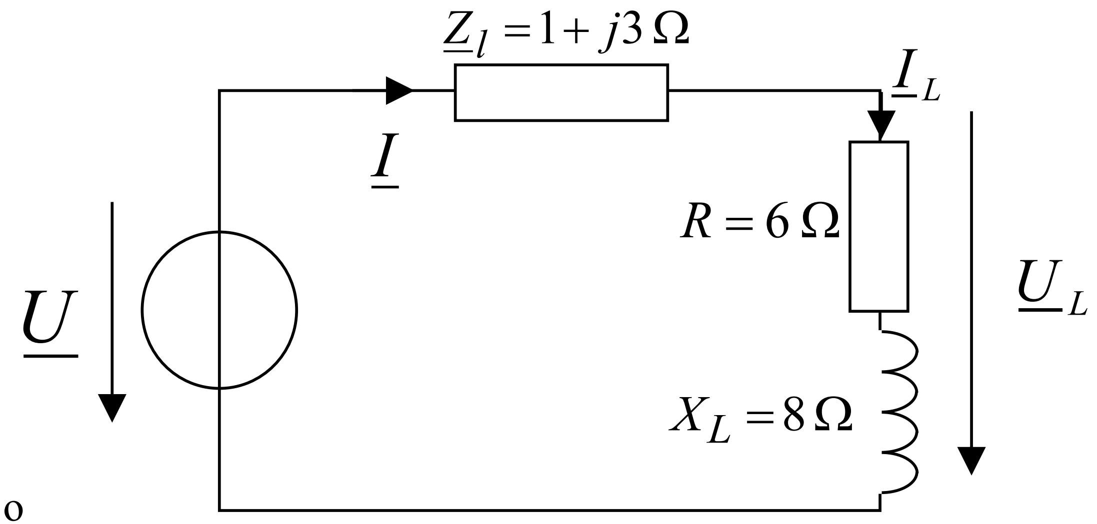
$$\underline{I}_L = \frac{250e^{j0^{o}}}{10e^{j53.1^{o}}} = 25e^{-j53.1^{o}} \ A$$

$$\underline{S} = 250e^{j0^{o}} 25e^{j53.1^{o}} = 6.25e^{j53.1^{o}} \ kVA$$

$$P = 6.25\cos 53.1^{o} = 3.75 \ kW$$

$$Q = 6.25\sin 53.1^{o} = 5.0 \ kvar$$

$$F_p = \cos 53.1^{o} = +0.6$$

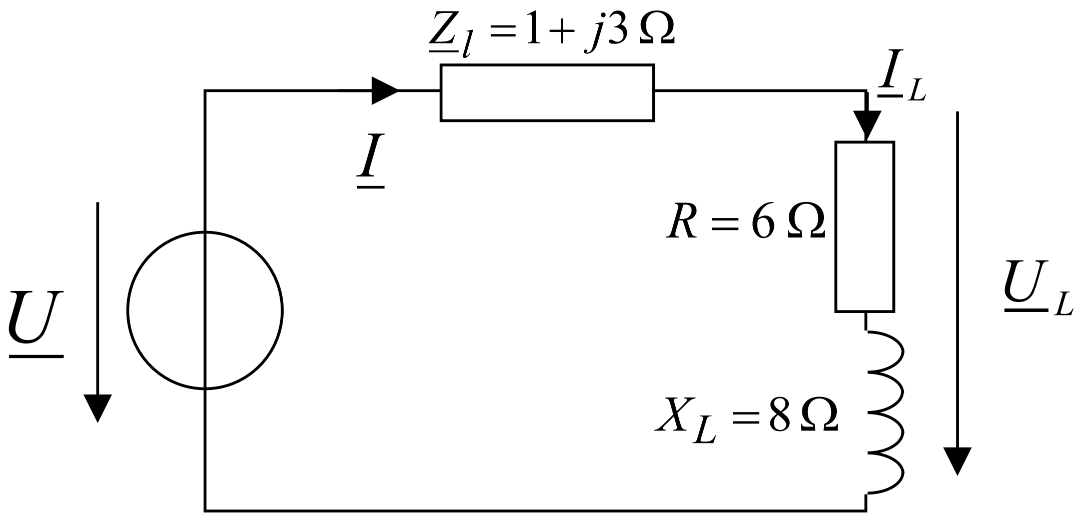


# Active and reactive powers: example

An industrial charge is represented by an impedance formed by the series setting of a resistor and an inductor. The voltage across the charge is $U_{ch}$= $250e^{j0}$ (V).

a) Calculate the current $I_{ch}$, Q, P, S and $F_p$ of the charge.

**b) Calculate the voltage of the source if the line connecting the source to the charge has a given impedance $Z_l$. Calculate the power lost in the line.**

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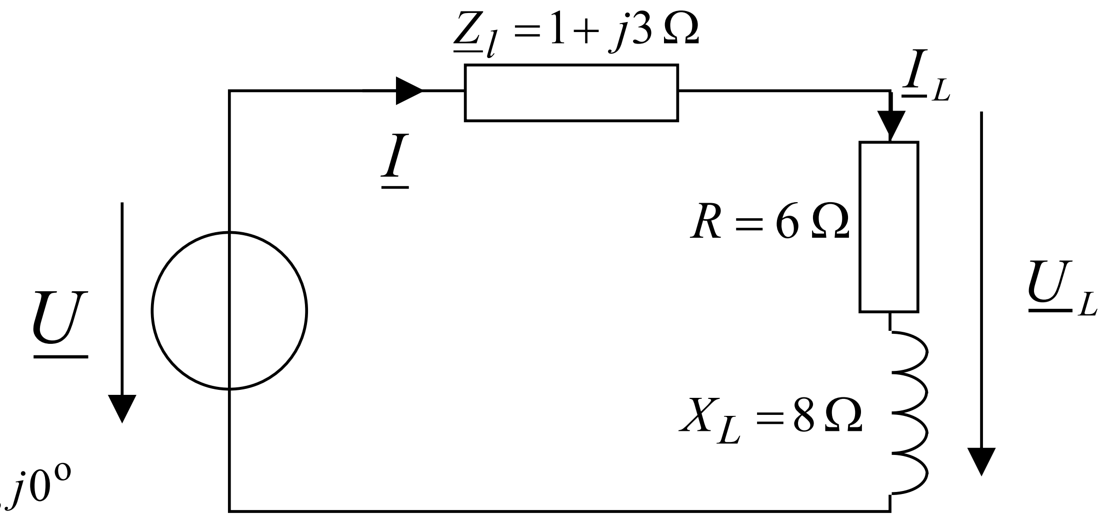
d) Calculate the new voltage of the source and the new power loss in the line.

# Solution (continued)

b)

$\underline{U} = \underline{Z}_l \underline{I}_L + \underline{U}_L$

$\underline{Z}_l = 1 + 3j = 3.16 e^{j71.57^o}\ \Omega$




$\underline{U} = 3.16 e^{j71.6^o}\, 25 e^{-j53.1^o} + 250 e^{j0^o}$

$= 79 e^{j18.5^o} + 250 e^{j0^o}$

$= 325.8 e^{j4.4^o}\quad V$

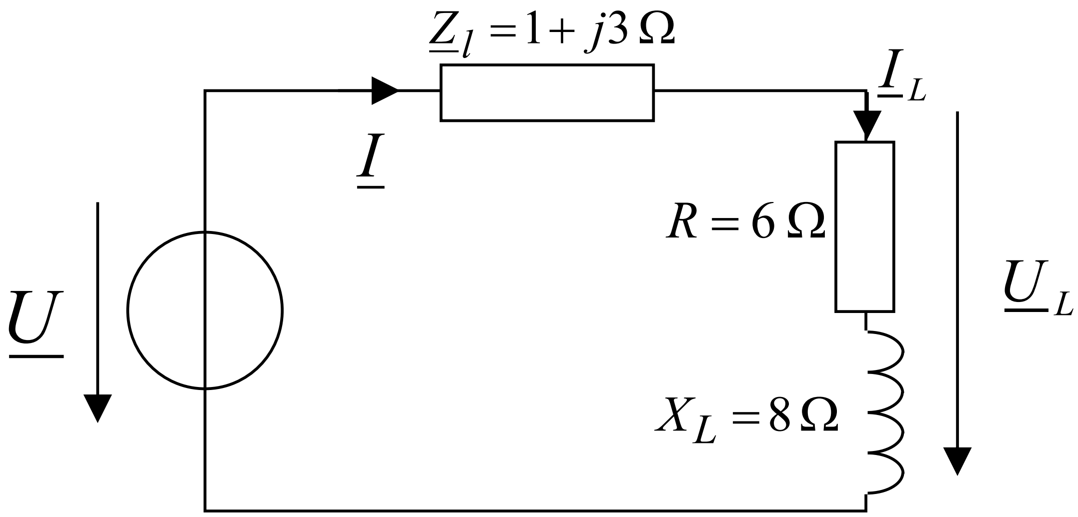
$\underline{P}_l = R_l I^2 = R_l I_L^2 = 625\, W$

# Active and reactive powers: example

An industrial charge is represented by an impedance formed by the series setting of a resistor and an inductor. The voltage across the charge is $\underline{U}_{ch}$= $250e^{j0}$ (V).

a) Calculate the current $\underline{I}_{ch}$, Q, P, S and $F_p$ of the charge.

b) Calculate the voltage of the source if the line connecting the source to the charge has a given impedance $\underline{Z}_l$. Calculate the power lost in the line.

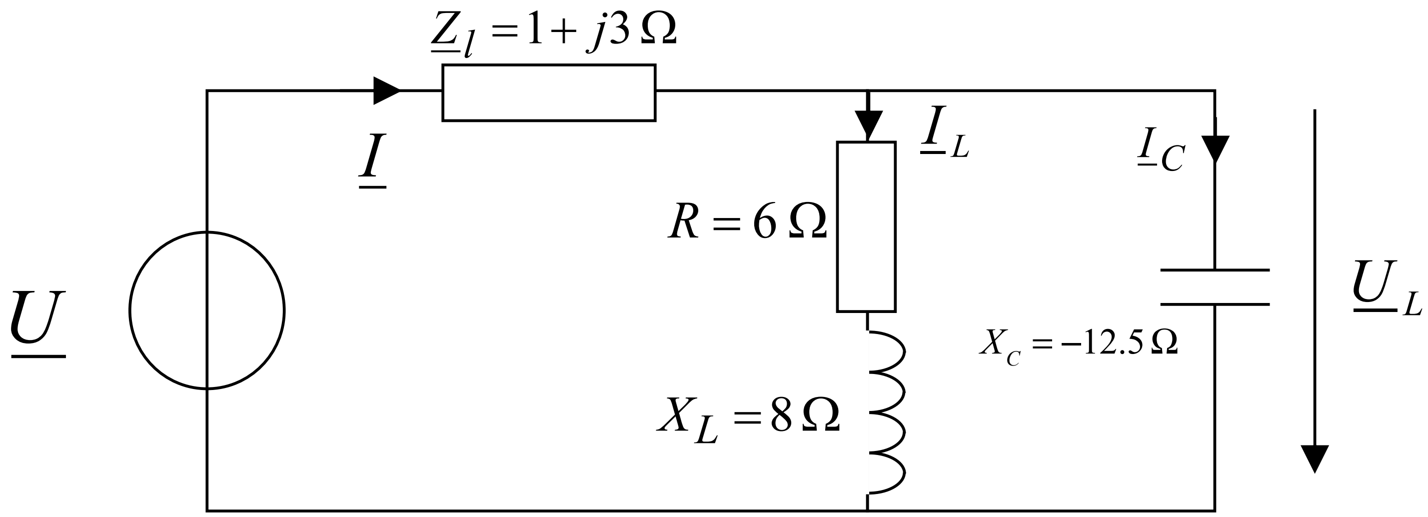



**c) If we add a capacitor of $X_C$ =-12.5 Ω in parallel, calculate the current taken by the C, the new current supplied by the source and $F_p$ of the entire charge and capacity (for the same voltage across the charge).**

d) Calculate the new voltage of the source and the new power loss in the line.

# Solution (continued)

c)




$$\underline{I}_C = \frac{\underline{U}_L}{\underline{Z}_C} = \frac{250}{12.5e^{-j90^o}} = 20e^{j90^o} \quad A$$

$$\underline{I} = \underline{I}_L + \underline{I}_C = 25e^{-j53.1^o} + 20e^{j90^o} = 15e^{j0^o} \quad A$$

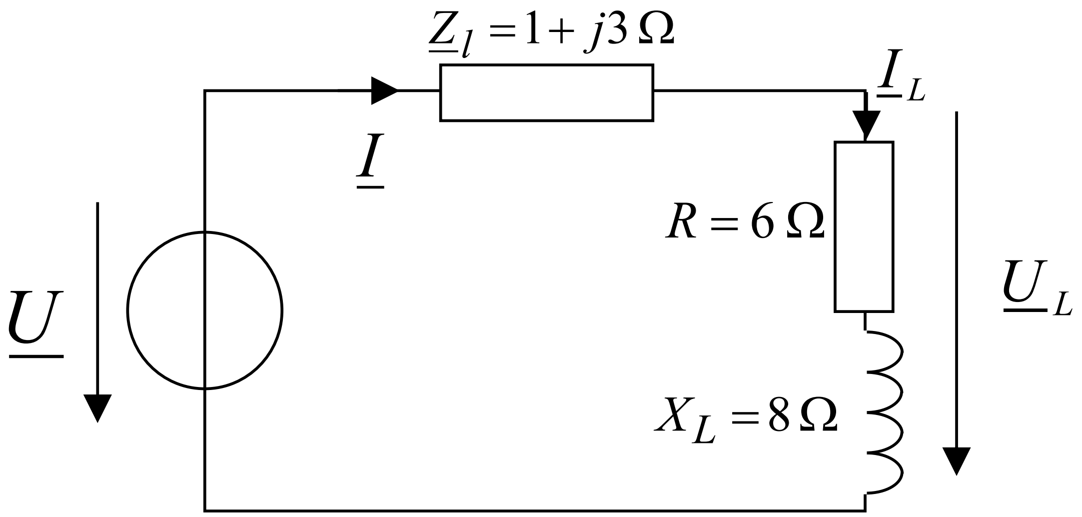
$$F_p = \cos 0^o = 1$$

# Active and reactive powers: example

An industrial charge is represented by an impedance formed by the series setting of a resistor and an inductor. The voltage across the charge is $\underline{U}_{ch}$= $250e^{j0}$ (V).

a) Calculate the current $\underline{I}_{ch}$, Q, P, S and $F_p$ of the charge.

b) Calculate the voltage of the source if the line connecting the source to the charge has a given impedance $\underline{Z}_l$. Calculate the power lost in the line.

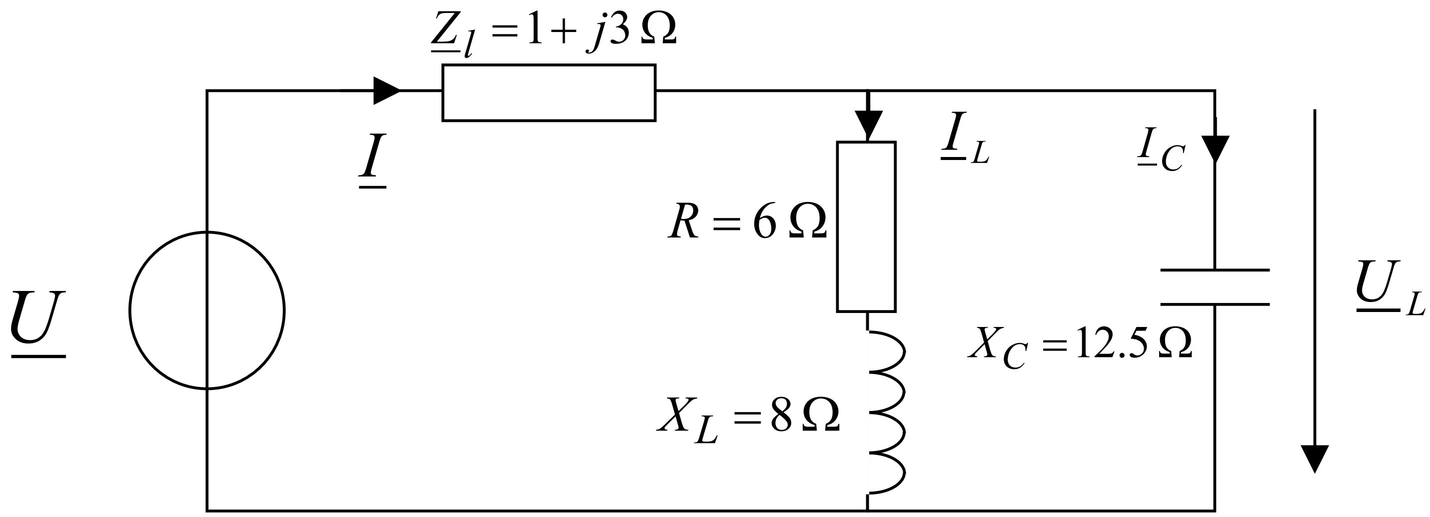



c) If we add a capacitor of $X_C$ =-12.5 Ω in parallel, calculate the current taken by the C, the new current supplied by the source and $F_p$ of the entire charge and capacity (for the same voltage across the charge).

**d) Calculate the new voltage of the source and the new power loss in the line.**

# Solution (continued)

d)




$$\underline{U} = 3.16e^{j71.6^o}\, 15e^{j0^o} + 250e^{j0^o}$$

$$= 47.4e^{j71.6^o} + 250e^{j0^o}$$

$$= 265 + j45 = 269e^{j10^o}\ V$$

$$\underline{P}_l = R_l I^2 = 225\, W$$