Final Exam

FIRST NAME : .... LAST NAME : ....

SCIPER : ....

Problem 1:

a) Derive the inverse Discrete-Time Fourier Transform (DTFT) of:

$$Y(\omega) = \frac{1}{(1-\alpha e^{-j\omega})^2} \qquad \text{for } \alpha \in (0,1)$$

You can only assume the knowledge of the inverse DTFT of $\frac{1}{(1-\alpha e^{-j\omega})}$ from the table.

Hint for b and c: *use the following property of partial fractions:* $\frac{1}{(a+x)(b+x)} = \frac{1}{b-a} \cdot \left[\frac{1}{a+x} - \frac{1}{b+x}\right]$

b) Consider the following causal discrete LTI system that is characterized by the difference equation with input signal $x[n]$ and output signal y[n]:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n] \tag{1}$$

Compute the impulse response of the system $h[n]$ and its corresponding DTFT $H(\omega)$. Is the system stable?

c) Consider the same LTI system given by Eq. 1. Find the output signal $y[n]$ of this system when we apply the following input signal:

$$x[n] = \left(\frac{1}{4}\right)^n u[n]. \tag{2}$$

where $u[n]$ is the unit step function.

Solution ***(6.0 points)***

1. *(2.0 points)* From the appendix 4.D we have the inverse DTFT of $Y(\omega)$ is given by:

$$x[n] = \alpha^n u[n] \tag{3}$$

where,

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{4}$$

with $|\alpha| < 1$

Moreover we have the relation:

$$\begin{aligned} nx[n] \xleftrightarrow{DTFT} & j\frac{dX(\omega)}{d\omega} \\ =& j\frac{-j\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} \\ =& \alpha e^{-j\omega} Y(\omega) \end{aligned}$$

Using linearity and the shift in time property we obtain:

$$\begin{aligned} y[n] &= \frac{1}{\alpha}(n+1)x[n+1] && (5) \\ &= (n+1)\alpha^n u[n+1] && (6) \\ &= (n+1)\alpha^n u[n] \quad \text{as, } y[-1] = 0 && (7) \\ & && (8) \end{aligned}$$

with $0 < |\alpha| < 1$

2. *(2.0 points)* Taking Fourier transform on both sides we can write:

$$\begin{aligned} H(\omega) = \frac{Y(\omega)}{X(\omega)} &= \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}} \\ &= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} \\ &= \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}} \end{aligned}$$

Therefore, the impulse response is given by

$$h[n] = 4 \cdot \left(\frac{1}{2}\right)^n u[n] - 2 \cdot \left(\frac{1}{4}\right)^n u[n]$$

Since any transfer function $h[n] = \alpha^n u[n]$ is stable for $|\alpha| < 1$, we conclude that the system is stable

3. *(2.0 points)* Fourier transform of $x[n] = \left(\frac{1}{4}\right)^n u[n]$ is given as $X(\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}}$. Therefore $Y(\omega)$ is given as:

$$\begin{aligned} Y(\omega) &= H(\omega) \cdot X(\omega) \\ &= \left[\frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right] \\ &= \frac{4}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} \\ &= \frac{8}{1-\frac{1}{2}e^{-j\omega}} - \frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} \end{aligned}$$

We can use differentiation property in Fourier space to calculate the inverse transform of the last term $\frac{1}{(1-\frac{1}{4}e^{-j\omega})^2}$.

$$y[n] = \left[8\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n\right]u[n]$$

Problem 2:

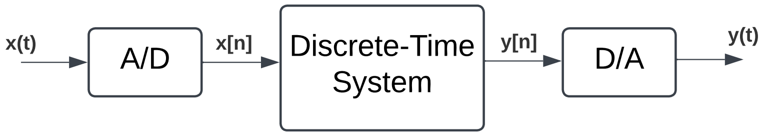

Figure 1: High-level diagram of a signal processing system

Figure 1 shows the high-level diagram of a signal processing system, consisting of analog-to-digital converter (A/D), discrete-time system, and digital-to-analog converter (D/A).

a) Given the time domain signal $x(t) = 2+5cos(800\pi t+\frac{\pi}{6})+3sin(500\pi t)$, find the continuous Fourier transform $X(w) = F\{x(t)\}$, and plot the magnitude of the spectrum.

b) Find the minimum sampling rate of A/D converter such that $x(t)$ can be fully recovered from $x[n]$.

c) Assume that the sampling rate of A/D converter is $f_s = 1000$ Hz. Give the expression of the discrete signal $x[n] = x(nT_s)$, where $T_s = \frac{1}{f_s}$.

d) The low-level diagram of a discrete-time system is given in Figure 2, consisting of three sub-blocks with frequency responses $F\{h_1[n]\} = H_1(w),\ F\{h_2[n]\} = H_2(w),\ F\{h_3[n]\} = H_3(w)$.

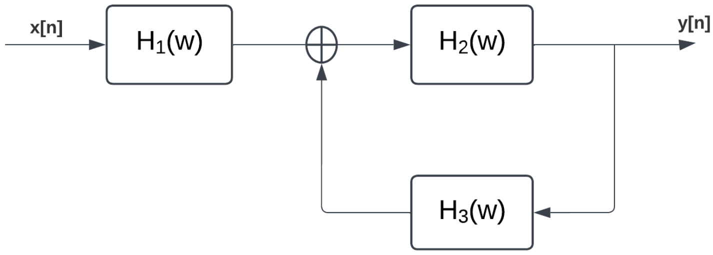

Figure 2: Low-level diagram of the discrete-time system

Find the end-to-end transfer function of this discrete-time system.

Solution ***(4.0 points)***

a) *(1.0 point)* From Appendix 4.B,

$$X(w) = F\{x(t)\}$$

$$X(w) = 4\pi\delta(w) + 5\pi[e^{j\frac{\pi}{6}}\delta(w-800\pi) + e^{-j\frac{\pi}{6}}\delta(w+800\pi)] + 3\frac{\pi}{j}[\delta(w-500\pi) - \delta(w+500\pi)]$$

b) *(1.0 point)* To be able to fully recover the continuous time signal $x(t)$ from its samples $x[n]$, $x(t)$ should be sampled at least at its Nyquist rate. Since the highest frequency of the signal $x(t)$ is $w_M = 800\pi$, by the sampling theorem,

$$w_s > 2w_M$$

$$w_s > 1600\pi$$

The minimum sampling rate of A/D converter should be $w = 1600\pi$ rad or $f = 800$ Hz.

c) *(1.0 points)*

$$x[n] = x(nT_s)$$

$$x[n] = 2 + 5\cos\left(\frac{800\pi n}{f_s} + \frac{\pi}{6}\right) + 3\sin\left(\frac{500\pi n}{f_s}\right)$$

$$x[n] = 2 + 5\cos\left(\frac{4\pi n}{5} + \frac{\pi}{6}\right) + 3\sin\left(\frac{\pi n}{2}\right)$$

d) *(1.0 point)*

$$H(w) = \frac{Y(w)}{X(w)}$$

Writing the input-output relation,

$$y[n] = [x[n] * h_1[n] + y[n] * h_3[n]] * h_2[n]$$

Taking the Fourier transform of both sides,

$$Y(w) = (X(w)H_1(w) + Y(w)H_3(w))H_2(w)$$

$$Y(w) = X(w)H_1(w)H_2(w) + H_2(w)H_3(w)Y(w)$$

$$(1 - H_2(w)H_3(w))Y(w) = H_1(w)H_2(w)X(w)$$

$$H(\omega) = \frac{Y(w)}{X(w)} = \frac{H_1(w)H_2(w)}{1 - H_2(w)H_3(w)}$$

Problem 3: Analysis of a PID

A PID controller is a feedback-based control loop mechanism commonly used to manage machines and processes that require continuous control and automatic adjustment.

Let $G(s)$ be the transfer function of the process we want to regulate and $C(s)$ the transfer function of the PID controller. The figure (3) illustrate how the feedback loop is made.

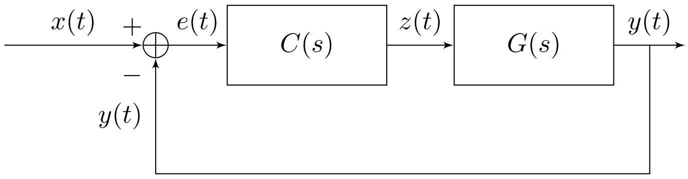

Figure 3: Illustration of a PID feedback control loop.

a) The PID controller time-domain equation is given by the following relation:

$$z(t) = k_p e(t) + k_i \int_{-\infty}^{t} e(\tau) d\tau + k_d \frac{d}{dt} e(t)$$

where k_p, k_i & k_d are real constants.

Give the corresponding transfer function $C(s) = \frac{Z(s)}{E(s)}$.

b) Suppose $k_p = 1, k_i = 2, k_d = 0$ and $G(s) = \frac{1}{s+1}$.

We obtain the following "open-loop" transfer function:

$$H_{op}(s) = G(s)C(s) = \frac{s+2}{s(s+1)} \tag{9}$$

i. Draw the pole-zero plot of this "open-loop" system.

ii. What are the possible Regions of Convergence (ROC's) of $H_{op}(s)$? Justify.

iii. Can the system describe by $H_{op}(s)$ be causal? And stable? Justify.

c) Suppose now that we have the following end-to-end transfer function:

$$H(s) = \frac{s+2}{(s^2 + 2s + 2)} \tag{10}$$

i. Draw the pole-zero plot of that system.

ii. What are the possible Regions of Convergence (ROC's) of $H(s)$? Justify.

iii. Suppose the system is causal, is it also stable? Justify.

d) Use the inverse Laplace transform to get $h(t)$.

Hint: Notice the similarity with the Laplace transform of one-sided Cosines ***and*** *Sines*

Solution ***(5.0 points)***

a) *(1.0 point)* Here we can use the Appendix 6.A and write $Z(s)$ in function of $E(s)$:

$$Z(s) = \left(k_p + \frac{k_i}{s} + sk_d\right) E(s) \tag{11}$$

We thus obtain:

$$C(s) = \frac{Z(s)}{E(s)} = \left(k_p + \frac{k_i}{s} + sk_d\right) \tag{12}$$

b) i. *(0.5 point)* Here we can already see that $s = -2$ is a zero. The poles are $s = 0$ and $s = -1$. The figure 4 shows the resulting pole-zero plot:

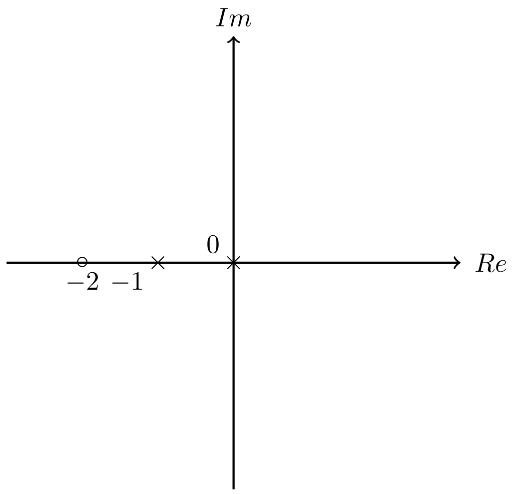

Figure 4: Pole-zero plot for $H_{op}(s)$

ii. *(0.5 point)* By looking at the pole-zero plot we can see the possible ROC:

- $Re(s) < -1$
- $-1 < Re(s) < 0$
- $Re(s) > 0$

iii. *(0.5 point)* Let us restate the following conditions for causality and stability. Assuming a rational transfer function

A system is

Causal	if and only if the region of convergence is the plane right of the rightmost pole.
Stable	if and only if the region of convergence contains the imaginary axis of the s-plane.
Causal and Stable	if and only if all poles lie in the left half of the s-plane and the region of convergence lies right of the rightmost pole.

In this case, we see that the system is causal but unstable.

c) i. *(0.5 point)* Here we can already see that $s = -2$ is a zero. To find the poles we have to find the roots of $s^2 + 2s + 2 = 0$ Which are $s_{1,2} = -1 \pm j$. The figure 5 shows the resulting pole-zero plot:

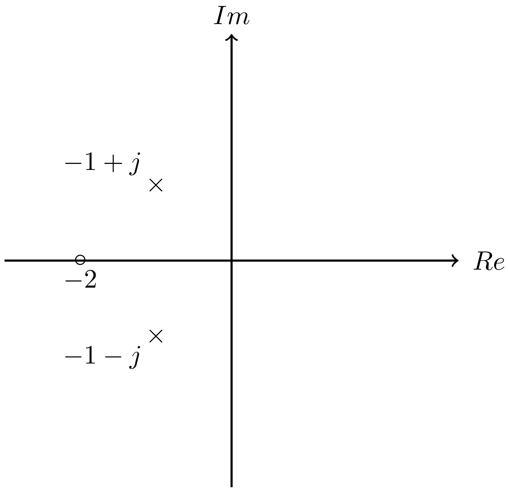

Figure 5: Pole-zero plot for $H(s)$

ii. *(0.5 point)* By looking at the pole-zero plot we can see the possible ROC:

- $Re(s) < -1$
- $Re(s) > -1$

iii. *(0.5 point)*

In this case, we see that the system is causal and stable because the ROC contains the imaginary axis $s = jw$.

d) *(1.0 point)* From Appendix 6.B we have the relation:

$$e^{-\alpha t}\cos(\omega_0 t)u(t) \overset{Laplace}{\longleftrightarrow} \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}, \quad \text{for } Re(s) > -Re(\alpha)$$
$$e^{-\alpha t}\sin(\omega_0 t)u(t) \overset{Laplace}{\longleftrightarrow} \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}, \quad \text{for } Re(s) > -Re(\alpha)$$

We can rewrite $H(s)$ to use these relations:

$$H(s) = \frac{s+2}{(s^2+2s+2)} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} \tag{13}$$

Here we identify $w_0 = \alpha = 1$ and knowing that the system is stable for the ROC $Re(s) > -Re(1)$. Hence:

$$h(t) = e^{-t}(\cos(t) + \sin(t)) \tag{14}$$