Homework 3, Computational Complexity 2025

The deadline is 23:59 on Friday 5 December. Please submit your solutions on Moodle. Typing your solutions using LaTeX is strongly encouraged. The problems are meant to be worked on in groups of 1–2 students. Please submit only one writeup per team. You are strongly encouraged to solve these problems by yourself. If you must, you may use books or online resources to help solve homework problems, but you must credit all such sources in your writeup and you must never copy material verbatim.

1 We say a CNF $\varphi = C_1 \wedge \cdots \wedge C_m$ is *Horn* if each clause C_i has at most one positive literal.

1a Show that $\textsc{HornSAT} = \{\langle \varphi \rangle : \varphi \text{ is Horn and satisfiable}\}$ is in P.

(*Hint: Note that if all clauses have at least two literals, then the all-*0 *assignment satisfies all clauses. What happens if we have singleton clauses?*)

1b Show that if φ is Horn and unsatisfiable, then it admits a polynomial-size Resolution refutation.

Solution:

1a Note that deciding whether φ is Horn can be done in linear time: just iterate over clauses and check that each clause contains at most one positive literal. Now, we describe an algorithm that decide whether $\varphi \in \text{SAT}$ provided φ is Horn. Observe that if each clause has two literals, then each clause has at least one negative literal so that the all-0 assignment satisfies φ and thus $\varphi \in \text{SAT}$. Suppose now that φ has a clause with a single literal $\overline{x_i}$ (the x_i case is analogous). Note that the only way to satisfy φ is to have $x_i = 0$. If x_i appears as another clause in φ, then $\varphi \notin \text{SAT}$. Else, we substitute $x_i = 0$ in φ and simplify to get φ' (e.g. a clause like $\overline{x_i} \vee x_j$ gets removed). Note that φ' is still horn but has less variable and we recurse on it. If the empty formula is reached, then $\varphi \in \text{SAT}$.

1b To ease the notation, we will see CNFs as sets of clauses. We show that any unsatisfiable Horn CNF $\mathcal{C}$ has a resolution refutation of size $\leq |\mathcal{C}|^2$ by strong induction on the number of clauses $|\mathcal{C}|$. The case $|\mathcal{C}| = 0$ is trivial because the empty CNF has a resolution refutation of size 0. The case $|\mathcal{C}| = 1$ cannot happen because that would imply that $\mathcal{C}$ is satisfiable. Fix now $\mathcal{C}$ with $|\mathcal{C}| \geq 2$. Because $\mathcal{C}$ is Horn and unsatisfiable, it has a clause of the form $\overline{x_i}$ (the case x_i is analogous). If $\mathcal{C}$ also has a clause of the form x_i, then a single resolution application is enough to prove that $\mathcal{C}$ is not satisfiable. If not, define $\mathcal{C}'$ as a copy of $\mathcal{C}$ where any clause where $\overline{x_i}$ appears is removed and any clause of the form $C = x_i \cup C'$ is replaced by C'. Observe that:

(a) $\mathcal{C}'$ is still Horn because applying the resolution rule reduce with a singleton clause yields a smaller clause.

(b) $\mathcal{C}'$ is still unsatisfiable. By construction, $\mathcal{C}'$ does not contain the variable x_i. Thus, if $\mathcal{C}'$ had a satisfying assignment α, then $\alpha \cup \{x_i = 0\}$ would be a satisfying assignment for $\mathcal{C}$ too.

We can obtain $\mathcal{C}'$ from $\mathcal{C}$ by applying the resolution rule at most $|\mathcal{C}|$ times. Note also that $|\mathcal{C}'| \leq |\mathcal{C}| - 1$ because $\overline{x_i} \in \mathcal{C} \setminus \mathcal{C}'$. Hence, we may use the induction hypothesis to get that $\mathcal{C}'$ has a resolution refutation of size $\leq |\mathcal{C}'|^2$ so that $\mathcal{C}$ admits a resolution refutation of size $\leq \mathcal{C} + (|\mathcal{C}| - 1)^2 \leq |\mathcal{C}|^2$, as desired.

If one cares about the bit complexity of the proof (i.e. the number of literals across the whole proof) note that this is also polynomialy bounded as every application of the resolution rule yields a smaller clause.

2 Let $F\colon \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ be a two-party communication problem where Alice gets $x \in \mathcal{X}$ and Bob gets $y \in \mathcal{Y}$. A set $S \subseteq \mathcal{X} \times \mathcal{Y}$ is called a 0-*fooling set* iff (i) $F(x,y) = 0$ for all $(x,y) \in S$, and (ii) for any two distinct $(x,y),(x',y') \in S$ we have $F(x,y') = 1$ or $F(x',y) = 1$. Let $\mathrm{fs}_0(F)$ denote the largest size $|S|$ of a 0-fooling set S for F.

2a Show that $\mathrm{N}_0(F) \geq \log \mathrm{fs}_0(F)$ for every F.

2b Show that $\mathrm{fs}_0(\mathrm{SI}_n) = 2^n$ where $\mathrm{SI}_n(x,y) \coloneqq \bigvee_{i=1}^n (x_i \wedge y_i)$ is the set-intersection problem.

Solution: See solutions to Final Exam 2024 on Moodle.

3 A *partition*[*] of $[n]$ is a collection $\mathcal{P}$ of non-empty subsets of $[n]$ such that for each $i \in [n]$ there is a unique $P \in \mathcal{P}$ with $i \in P$. For partitions $\mathcal{P}$, $\mathcal{Q}$ we say that $\mathcal{P}$ *refines* $\mathcal{Q}$, denoted $\mathcal{P} \sqsubseteq \mathcal{Q}$, if for every $P \in \mathcal{P}$ there is some $Q \in \mathcal{Q}$ such that $P \subseteq Q$. For example, for $n = 6$, we have that $\{\{1,4\},\{2\},\{3,5\},\{6\}\}$ refines $\{\{1,3,4,5\},\{2,6\}\}$. Define the two-party *refinement problem* by

$$\textsc{Ref}_n(\mathcal{P},\mathcal{Q}) \;=\; \begin{cases} 1 & \text{if } \mathcal{P} \sqsubseteq \mathcal{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Show a lower bound $\mathrm{R}^{\mathrm{cc}}_{1/3}(\textsc{Ref}_n) \geq \Omega(n)$ by a reduction from set-intersection (or its complement).

(*Fun fact: Only a trivial upper bound is known,* $\mathrm{R}^{\mathrm{cc}}_{1/3}(\textsc{Ref}_n) \leq O(n \log n)$*, which is just the length of the input. It is an open problem to close this logarithmic gap.*)

Solution: We present a reduction to $\overline{\mathrm{SI}}$, the complement of the set-intersection problem, i.e. $\overline{\mathrm{SI}}(x,y) = 1$ if and only if $x \cap y = \emptyset$. Since randomized communication complexity is closed under complement, we have that $\mathrm{R}^{\mathrm{cc}}_{1/3}(\overline{\mathrm{SI}}_n) \geq \Omega(n)$. The reduction idea is simple: note that $x \cap y = \emptyset$ if and only if $y \subseteq \overline{x}$ (the complement of the set described by x). This is not yet a set-refinement instance though, because y and $\overline{x}$ may not partition $[n]$. To fix this, we extend $[n]$ by introducing one more gadget integer $M \coloneqq n + 1$.

We describe how to reduce $\overline{\mathrm{SI}}_n$ to $\textsc{Ref}_M$. On input x, Alice creates the partition $\mathcal{P}$ consisting of $\overline{x} \cup \{M\}$ and the rest of the elements of $[M]$ as singleton. Bob creates the partition $\mathcal{Q}$ consisting of $y \cup \{M\}$ and the rest as singletons (see Figure 1 for an example). Note that by construction, this is a valid set-refinement instance. Suppose now that $\mathcal{Q} \sqsubseteq \mathcal{P}$, then in particular $y \cup \{M\} \subseteq \overline{x} \cup \{M\}$ thus $x \cap y = \emptyset$. On the other hand, if $\mathcal{Q} \not\sqsubseteq \mathcal{P}$, then $y \cup \{M\} \not\subseteq \overline{x} \cup \{M\}$ so that $x \cap y \neq \emptyset$.

[*] https://en.wikipedia.org/wiki/Partition_of_a_set

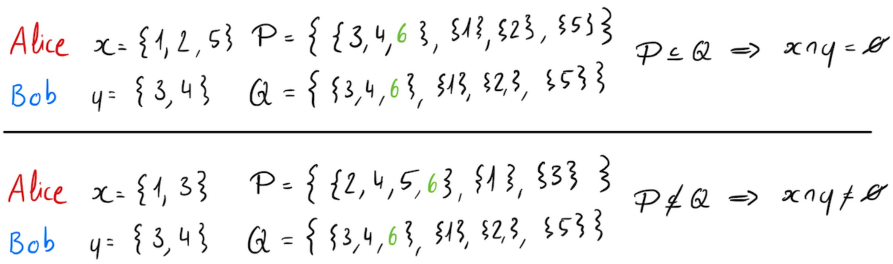

Figure 1. The reduction from $\overline{\mathrm{SI}}_5$ to $\textsc{Ref}_6$.