



Final Exam, Computational Complexity 2024

- Books, notes, cellphones, etc. are **not** allowed.
- Your explanations and proofs should be clear enough and in sufficient detail so that they are easy to understand and have no ambiguities.
- You are allowed to freely refer to material covered in lectures/exercises.
- **Do not touch until the start of the exam.**

Good luck!

Name: _____

N° Sciper: _____

1 (15 pts) **Quick-fire round.** Consider the following statements.

1. $\mathbf{BPP} \subseteq \mathbf{NP}$
2. $\mathbf{L} \neq \mathbf{PSPACE}$
3. $\mathbf{PSPACE} = \mathbf{NPSPACE}$
4. If $\mathbf{P} = \mathbf{NP}$, then $\mathbf{PH} = \mathbf{P}$.
5. If $\mathbf{PH} = \mathbf{PSPACE}$, then \mathbf{PH} collapses.
6. If $A \in \mathbf{P}^B$, then $A \leq_p B$.
7. If $\mathbf{NP} = \mathbf{coNP}$, then $\mathbf{NP}^A = \mathbf{coNP}^A$ for all oracles A .
8. Savitch's algorithm solves st -connectivity in polynomial time.
9. Language $\{\langle G \rangle : G \text{ is a graph with vertex cover of size } 2024\}$ is \mathbf{NP} -complete.
10. Language $\{\langle \varphi \rangle : \varphi \text{ is a CNF that has at least } 2024 \text{ satisfying assignments}\}$ is \mathbf{NP} -complete.
11. Language $\{\langle \varphi \rangle : \varphi \text{ is a satisfiable CNF whose clauses have } \textit{two} \text{ literals}\}$ is \mathbf{NP} -complete.
12. Language $\{\langle C \rangle : C \text{ is a boolean circuit such that } C(x) = 1 \text{ for all } x\}$ is \mathbf{NP} -complete.
13. Every CNF formula admits a Resolution refutation.
14. Every CNF formula φ can be equivalently written as a DNF formula ψ , that is, $\varphi(x) = \psi(x)$ for all x .
15. There is a *monotone* boolean function f whose decision tree depth complexity is larger than its certificate complexity, that is, $D(f) > C(f)$.

For each statement, write down one of

- **T** if the statement is known to be true.
- **F** if the statement is false *or not known to be true*. E.g., both $\mathbf{P} = \mathbf{NP}$ and $\mathbf{P} \neq \mathbf{NP}$ should be marked **F**.
- or leave your answer empty.

A correct **T/F** answer is worth +1 point, an incorrect answer is worth –1 point, and an empty answer is worth 0 points.

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2 (15 pts) **Problem Garden.** For each problem below, classify it into the smallest possible complexity class (as seen in the course). You do not need to justify your answers (but you can!).

1. **Input:** Graph G and integer k . **Output:** YES iff the largest clique in G is of size k .
2. **Input:** A DNF formula φ and a unary integer 1^k . **Output:** YES iff there is a CNF formula with at most k clauses that is equivalent to φ .
3. **Input:** Bipartite graph G . **Output:** YES iff G contains a perfect matching.
4. **Input:** Boolean circuit C with $2n$ input bits. **Output:** YES iff there is a directed path from 0^n to 1^n in the directed graph $G = (\{0, 1\}^n, E)$ defined by $(u, v) \in E$ iff $C(u, v) = 1$.
5. **Input:** Undirected graph G . **Output:** YES iff G contains a cycle of odd length.

- 3** (15 pts) **NP-completeness.** Suppose $\mathcal{S} = \{S_1, \dots, S_n\}$ is a family of subsets of $\{1, 2, \dots, n\}$. We say that k distinct sets $S_{i_1}, \dots, S_{i_k} \in \mathcal{S}$ form a *packing of size k* if the sets are pairwise disjoint, that is, $S_{i_j} \cap S_{i_{j'}} = \emptyset$ for $j \neq j'$. Show that the following problem is **NP**-complete:

$$\text{SETPACKING} = \{ \langle \mathcal{S}, k \rangle : \text{Family } \mathcal{S} \text{ contains a packing of size } k \}.$$

- 4 (15 pts) **Oracles.** A *k*-query oracle Turing machine is an oracle Turing machine that is permitted to make at most *k* queries on each input. Define $\mathbf{P}^{A,k}$ to be the collection of languages that are decidable by polynomial-time *k*-query oracle Turing machines with an oracle for *A*. Note that

$$\mathbf{NP} \cup \mathbf{coNP} \subseteq \mathbf{P}^{\text{SAT},1}.$$

Show that, if $\mathbf{NP} \neq \mathbf{coNP}$, then the above inclusion is strict, that is, $\mathbf{NP} \cup \mathbf{coNP} \neq \mathbf{P}^{\text{SAT},1}$.

- 5** (20 pts) **Decision trees.** Let $f: \{0, 1\}^n \rightarrow \{0, 1\}$ be a boolean function and let $\text{XOR}_2: \{0, 1\}^2 \rightarrow \{0, 1\}$ denote the 2-bit parity function ($\text{XOR}_2(x_1, x_2) = 1$ iff $x_1 + x_2$ is odd). Define the composed function $f \circ \text{XOR}_2$ by $(f \circ \text{XOR}_2)(x, y) := f(\text{XOR}_2(x_1, y_1), \dots, \text{XOR}_2(x_n, y_n))$ where $x, y \in \{0, 1\}^n$.
- 5a** Show that for every f , the function $f \circ \text{XOR}_2$ requires decision trees of *size* at least $2^{D(f)}$ (where, as usual, $D(f)$ denotes the decision tree *depth* complexity of f).
- 5b** Show that **5a** fails if we use the AND_2 function in place of XOR_2 . That is, exhibit some n -bit function f such that $f \circ \text{AND}_2$ admits a decision tree of size less than $2^{D(f)}$.

- 6** (20 pts) **Communication complexity.** Let $F: \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ be a two-party communication problem where Alice gets $x \in \mathcal{X}$ and Bob gets $y \in \mathcal{Y}$. A set $S \subseteq \mathcal{X} \times \mathcal{Y}$ is called a *0-fooling set* iff (i) $F(x, y) = 0$ for all $(x, y) \in S$, and (ii) for any two distinct $(x, y), (x', y') \in S$ we have $F(x, y') = 1$ or $F(x', y) = 1$. Let $\text{fs}_0(F)$ denote the largest size $|S|$ of a 0-fooling set S for F .
- 6a** Show that $N_0(F) \geq \log \text{fs}_0(F)$ for every F . (Here $N_0(F) = N_1(\neg F)$ is the co-nondeterministic communication complexity.)
- 6b** Show that $\text{fs}_0(\text{SI}_n) = 2^n$ where $\text{SI}_n(x, y) := \bigvee_{i=1}^n (x_i \wedge y_i)$ is the set-intersection problem.

