Exercise IX, Computational Complexity 2025

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

Resolution

1 Prove that tree-like Resolution (and hence Resolution, too) is *complete*: If φ is an unsatisfiable CNF, then there exists some tree-like Resolution refutation of φ.

(*Hint: For the easiest proof, use the equivalence between tree-like Resolution and decision trees solving* $\textsc{Search}(\varphi)$.)

2 Prove the lemma for *Tree–Adversary games* from the lecture. Namely, prove that if there exists an Adversary strategy for $\textsc{Search}(\varphi)$ that scores at least r points against any Tree strategy, then any decision tree solving $\textsc{Search}(\varphi)$ has size at least 2^r.

(*Hint: Prove the contrapositive. Given a decision tree, consider the Tree strategy that, when Adversary leaves the choice of value of* x_i *to Tree, it chooses the smaller subtree.*)

3 Recall that the n-bit $\textsc{Or}_n\colon \{0,1\}^n \to \{0,1\}$ has a decision tree of size $O(n)$. Let us modify $\textsc{Or}_n$ slightly by replacing each of its input variables with a 2-bit $\textsc{And}_2\colon \{0,1\}^2 \to \{0,1\}$. Namely, denote by $\textsc{Or}_n \circ \textsc{And}_2$ the function that on a $2n$-bit input $(x,y) \in \{0,1\}^n \times \{0,1\}^n$ outputs

$$(\textsc{Or}_n \circ \textsc{And}_2)(x,y) \;\coloneqq\; \textsc{Or}_n(\textsc{And}_2(x_1,y_1),\ldots,\textsc{And}_2(x_n,y_n)).$$

Show that any decision tree for $\textsc{Or}_n \circ \textsc{And}_2$ requires size 2^n.

4 The *width* of a Resolution refutation $\pi = (C_1,\ldots,C_s)$ is the maximum width $|C_i|$ of any clause C_i appearing in the proof.

(a) Show that if a CNF formula φ with n variables admits a width-w refutation, then it also admits one of size $s \le n^{O(w)}$.

(b) Given a formula φ and a width parameter w, show that one can find a width-w refutation of φ (if one exists) in time $n^{O(w)}$.

(This exercise shows that bounded-width Resolution is polynomial-time *automatable*; that is, short proofs can be found efficiently.)

5 Sometimes (for convenience) one allows an additional *weakening rule* in Resolution: From any clause A this rule allows to derive the clause $A \vee B$ where B is an arbitrary clause. Show that allowing this rule does not add power to Resolution: If a CNF formula φ has a size-s refutation in Resolution-with-weakening, then φ also has a size-s refutation in (usual) Resolution.