Exercise VIII, Computational Complexity 2025

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

Decision trees

1 Define $\textsc{Sink}\colon \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ by interpreting the input $x \in \{0,1\}^{\binom{n}{2}}$ as a labelling of the edges of the complete graph on n vertices such that the bit $x_e \in \{0,1\}$ for edge $e = \{u,v\}$ defines an *orientation* of e, either (u,v) or (v,u). Thus, an input x defines a directed graph $G_x = (V, E_x)$. We define $\textsc{Sink}(x) = 1$ iff the graph G_x has a *sink*, that is, some node $v^* \in V$ such that $(u, v^*) \in E_x$ for all $u \neq v^*$. Show that $\mathrm{D}(\textsc{Sink}) = \Theta(n)$.

2 Suppose $f\colon \{0,1\}^n \to \{0,1\}$ is *monotone*, meaning that if $x \leq y$ (i.e., $x_i \leq y_i$ for all $i \in [n]$), then $f(x) \leq f(y)$. Show that $\mathrm{s}(f) = \mathrm{C}(f)$.

3 In the lecture we saw that if there exists an Adversary strategy for answering $k-1$ many queries to the input variables of a function f such that the value of f remains undetermined, then $\mathrm{D}(f) \geq k$. Prove the converse: if $\mathrm{D}(f) \geq k$, then there exists an Adversary strategy fooling any $k-1$ query algorithm.

4 A DNF formula $F = T_1 \vee \cdots \vee T_m$ is said to be *unambiguous* if for any input x, at most one of the terms T_i evaluate to true, $T_i(x) = 1$. Define $\mathrm{UC}_1(f)$ as the least k such that f can be written as an unambiguous k-DNF. Prove that $\mathrm{D}(f) \leq \mathrm{UC}_1(f)^2$.

(*Hint: Use a similar idea as in the proof that* $\mathrm{D}(f) \leq \mathrm{C}(f)^2$.)

Definition 0.1 *For $x \in \{0,1\}^n$ and $S \subseteq [n]$, let x^S denote the point in $\{0,1\}^n$ obtained by flipping all of the bits x_i such that $i \in S$.*

The sensitivity of $f\colon \{0,1\}^n \to \{0,1\}$ at x, denoted $s_f(x)$, is the number of points y that differ from x in exactly one bit and satisfy $f(x) \neq f(y)$. The sensitivity $\mathrm{s}(f)$ *of f is the maximum over all $x \in \{0,1\}^n$ of* $\mathrm{s}(f,x)$.

The block sensitivity of $f\colon \{0,1\}^n \to \{0,1\}$ at x, denoted $\mathrm{bs}(f,x)$*, is the maximum number of disjoint subsets of the coordinates $B_1, B_2, \ldots, B_k \subseteq [n]$ such that $f(x) \neq f(x^{B_j})$ for all $j \in [k]$. The block sensitivity* $\mathrm{bs}(f)$ *is the maximum value of* $\mathrm{bs}(f,x)$ *over all $x \in \{0,1\}^n$.*

5 *Sensitivity vs. block-sensitivity.* Say that an n-bit string x is *paired* if there exists an $i \in [n-1]$ such that $x_i = x_{i+1} = 1$ and $x_j = 0$ for all other $j \in [n] \setminus \{i, i+1\}$. Define a boolean function $f\colon \{0,1\}^{n^2} \to \{0,1\}$ by interpreting the input $x \in \{0,1\}^{n^2}$ as an n-by-n boolean matrix and setting $f(x) = 1$ iff there exists a row of x that is paired. Prove that f has sensitivity $\mathrm{s}(f) \leq O(n)$ but quadratically larger block-sensitivity $\mathrm{bs}(f) \geq \Omega(n^2)$.