Exercise IV, Computational Complexity 2025

These exercises are for your own benefit. Feel free to collaborate and share your answers with other students. Solve as many problems as you can and ask for help if you get stuck for too long. Problems marked * are more difficult but also more fun :).

Diagonalisation and Oracles

1 Show that the problem

$$\textsc{ExpSim} = \{\langle M, x, 1^t \rangle : M \text{ is a TM, and } M(x) \text{ outputs 1 in} \leq 2^t \text{ steps}\}$$

is complete (under poly-time reductions) for the exponential-time class

$$\mathsf{EXP} = \textstyle\bigcup_{k \in \mathbb{N}} \mathsf{TIME}(2^{n^k}).$$

Namely, show that $\textsc{ExpSim} \in \mathsf{EXP}$ and that for every $A \in \mathsf{EXP}$ we have $A \leq_p \textsc{ExpSim}$.

2 Show that $\mathsf{TIME}(n)$ is not closed under poly-time reductions. That is, exhibit languages A and B such that $A \leq_p B$, $B \in \mathsf{TIME}(n)$, but $A \notin \mathsf{TIME}(n)$.

(Hint: Use the Time Hierarchy theorem.)

3 Show that the following problem is in $\mathsf{P}^{\mathrm{SAT}}$:

$$\textsc{MaxIS} = \{\langle G, k \rangle : G \text{ is a graph whose largest independent set is of size exactly } k\}$$

Do you think $\textsc{MaxIS}$ is in NP? What about coNP?

4 Show that the NP vs. coNP question relativises both ways. That is,

(a) There is an oracle A relative to which $\mathsf{NP}^A = \mathsf{coNP}^A$.
(b) There is an oracle B relative to which $\mathsf{NP}^B \neq \mathsf{coNP}^B$. Namely, consider the language

$$L_B = \{1^n : \forall x \in \{0,1\}^n, x \in B\}$$

and show that

- $L_B \in \mathsf{coNP}^B$ for every B.
- (*) $L_B \notin \mathsf{NP}^B$ for some B.

(Hint: Fix a poly-time nondeterministic oracle machine M^B. Can you define B for strings of large length n so that M^B reports the wrong answer for "$1^n \in L_B$"?)

5 (*) Call a language $A \subseteq \{0,1\}^*$ *sparse* if it contains only polynomially many strings of length n, that is, there is some $k \in \mathbb{N}$ such that $|A \cap \{0,1\}^n| \leq n^k$ for all n. Show that

$$L \in \mathsf{P/poly} \quad \text{iff} \quad L \in \mathsf{P}^A \text{ for some sparse } A.$$