CS-472: Design Technologies for Integrated Systems

Exercise Problem Set 7 Solution — Date: 04/12/2025

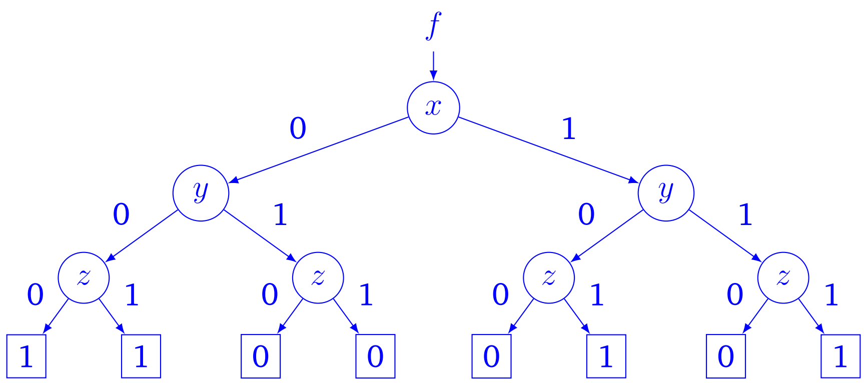
Topics: Binary Decision Diagrams (cf. slide set 10), Algebraic methods (cf. slide set 11)

Problem 1

A Boolean function $f(x, y, z)$ is given as the following truth table.

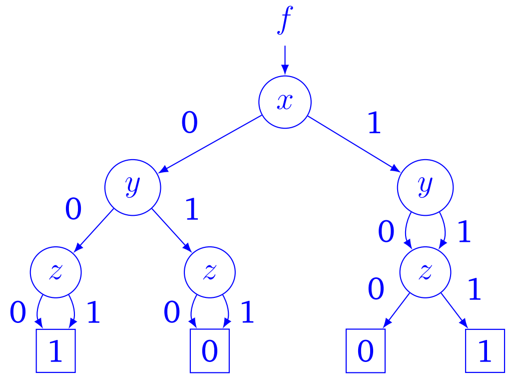
x	y	z	f
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

(a) Draw the complete, **non-reduced** BDD with the variable order $x < y < z$.

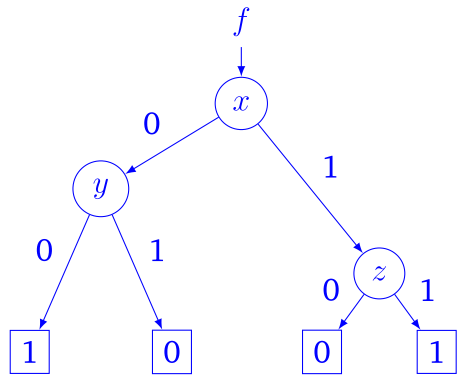


(b) Apply the reduction rules.

1. Merge equivalent subtrees:



2. Remove node with identical children:



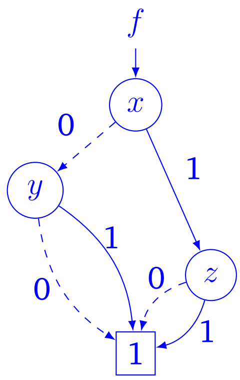
In software implementation, we usually also merge the constant 0 and 1 nodes, i.e., there are only one 0 node and one 1 node. Here, for clearer drawings, we keep them separated.

(c) Derive a SOP from the BDD.

Ans: $f = \bar{x}\bar{y} + xz$. (Hint: enumerate all paths leading to 1.)

(d) Transform the BDD to use complemented edges.
Rules:

1. You have only the constant 1 node, no constant 0.
2. The 1-edge (THEN-child) is never complemented – use the transformation in slide pp.45 to avoid it.

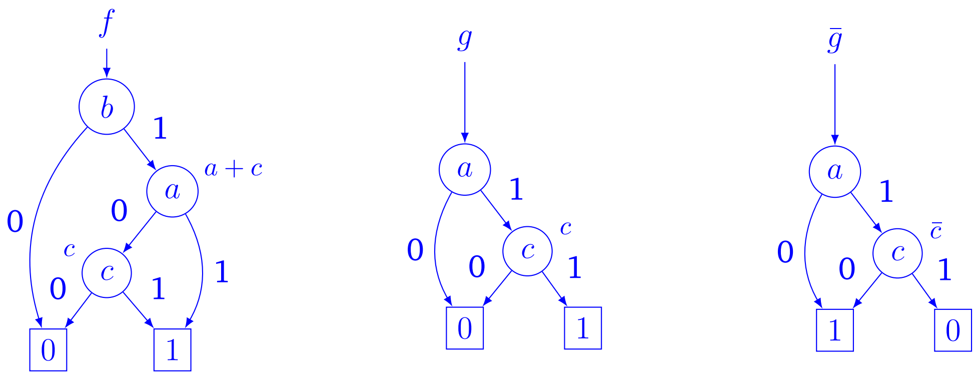


Note: We use dashed lines to represent complemented edges.

Problem 2

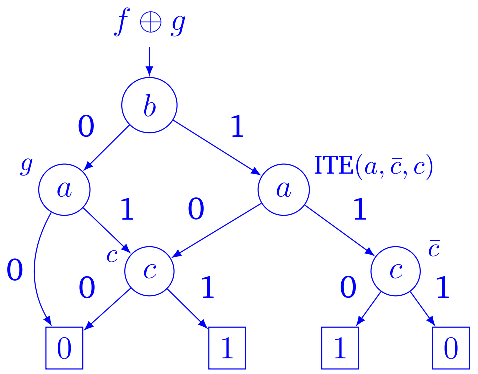
Consider the functions $f = ab + bc$ and $g = ac$.

(a) Draw the BDDs (reduced and ordered) for f, g and $\bar{g}$ (select a variable order that minimizes the BDDs).
Observe that $f = b(a + c)$. — b is more "important" than a and c, so we put b on top.



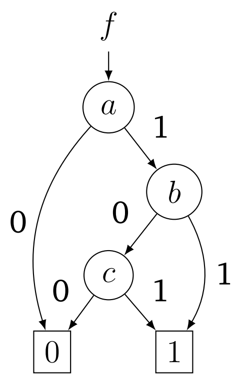
(b) Use the ITE operator to compute $f \oplus g$ and draw the BDD of $f \oplus g$.
Rewrite $f \oplus g$ using ITE and follow the algorithm on slides pp. 31:

$$\begin{aligned}
f \oplus g &= \text{ITE}(f, \bar{g}, g) \\
&= \text{ITE}(b, \quad \text{ITE}(f_b, \bar{g}_b, g_b), \quad \text{ITE}(f_{\bar{b}}, \bar{g}_{\bar{b}}, g_{\bar{b}})) \\
&= \text{ITE}(b, \quad \text{ITE}(a + c, \bar{g}, g), \quad \text{ITE}(0, \bar{g}, g)) \\
&= \text{ITE}(b, \quad \text{ITE}(a, \text{ITE}((a + c)_a, \bar{g}_a, g_a), \text{ITE}((a + c)_{\bar{a}}, \bar{g}_{\bar{a}}, g_{\bar{a}})), \quad g) \\
&= \text{ITE}(b, \quad \text{ITE}(a, \text{ITE}(1, \bar{c}, c), \text{ITE}(c, 1, 0)), \quad g) \\
&= \text{ITE}(b, \quad \text{ITE}(a, \bar{c}, c), \quad g)
\end{aligned}$$



Problem 3

Given the following zero-suppressed BDD (ZDD): *cf:* Slide pp.47-49.



(a) Write down the item set represented by the ZDD.
Hint: Each path leading to 1 is an item.
Ans: $\{ab, ac\}$, or some people prefer to write as a set of sets: $\{\{a, b\}, \{a, c\}\}$

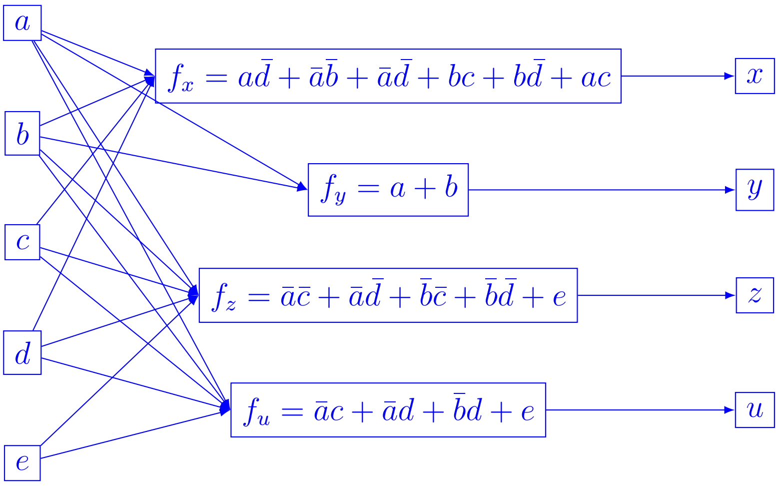
(b) Give an SOP of the characteristic function f. (Verify f with the item set!)
Hint: Each minterm of the characteristic function is an item in the item set.
Ans: $f = ab\bar{c} + a\bar{b}c$

Problem 4

Consider the following relations among Boolean variables $a, b, c, d, e, x, y, z, u$:

$$
\begin{aligned}
x &= a\bar{d} + \bar{a}\bar{b} + \bar{a}\bar{d} + bc + b\bar{d} + ac \\
y &= a + b \\
z &= \bar{a}\bar{c} + \bar{a}\bar{d} + \bar{b}\bar{c} + \bar{b}\bar{d} + e \\
u &= \bar{a}c + \bar{a}d + \bar{b}d + e
\end{aligned}
$$

(a) Draw the logic network, where inputs are $\{a, b, c, d, e\}$, outputs are $\{x, y, z, u\}$, and each node of the network can compute arbitrary function.

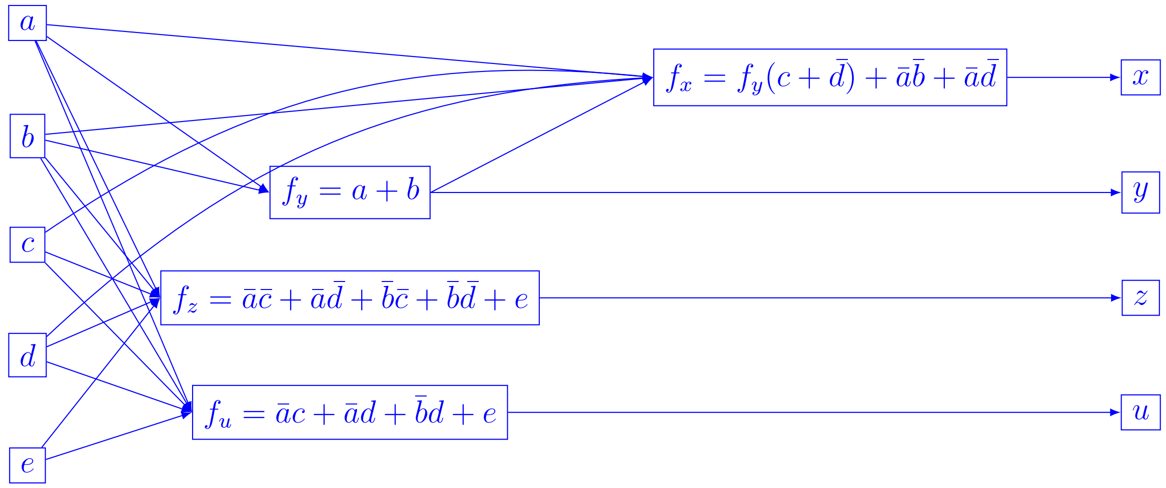


(b) Perform the algebraic division of f_x/f_y (f_x denotes the Boolean function of x in terms of a, b, c, d and e).
cf: Algorithm in textbook pp. 362.
Ans:

$$
\begin{aligned}
&A = \{a\bar{d}, \bar{a}\bar{b}, \bar{a}\bar{d}, bc, b\bar{d}, ac\} \\
&B = \{a, b\} \\
&\quad i = 1 : C_1^B = a. \;\; D = \{a\bar{d}, ac\}, D_1 = \{\bar{d}, c\} = Q. \\
&\quad i = 2 : C_2^B = b. \;\; D = \{bc, b\bar{d}\}, D_2 = \{c, \bar{d}\}, Q = \{c, \bar{d}\} \\
&R = \{\bar{a}\bar{b}, \bar{a}\bar{d}\} \\
\therefore\ & f_x = f_y(c + \bar{d}) + \bar{a}\bar{b} + \bar{a}\bar{d}
\end{aligned}
$$

Note that in algebraic division, a and $\bar{a}$, etc., are treated as unrelated symbols.

(c) Substitute y in f_x and redraw the network.



Problem 5

For the following functions:

$$F = abc + abd + c\bar{d}$$
$$G = \bar{b}d + a\bar{b} + \bar{b}c$$

(a) Compute all kernels and co-kernels of F and G.
cf: Algorithm in textbook pp. 369; slide 11 pp. 24.

sup(F)	a	b	c	d	$\bar{d}$
$j =$	1	2	3	4	5

KERNELS$(F, 1)$:
$i = 1$: $x_i = a$, $C = \{a, b\}$, $F/(ab) = c + d$, KERNELS$(c + d, 2)$= $\{c + d\}$
$i = 2$: $x_i = b$, $C = \{a, b\}$, C has $x_k = a$ where $k < i$, skip
$i = 3$: $x_i = c$, $C = \{c\}$, $F/c = ab + \bar{d}$, KERNELS$(ab + \bar{d}, 4)$= $\{ab + \bar{d}\}$
$i = 4$: $x_i = d$, CUBES(F, d)= $\{abd\}$, skip
$i = 5$: $x_i = \bar{d}$, CUBES$(F, \bar{d})$= $\{c\bar{d}\}$, skip
F is cube-free, so F itself is also a kernel with co-kernel being 1.
Ans:

$$K(F) = \{ \quad c + d, \quad ab + \bar{d}, \quad abc + abd + c\bar{d}\}$$
$$CoK(F) = \{ \quad ab, \quad c, \quad 1\}$$

sup(G)	$\bar{b}$	d	a	c
$j =$	1	2	3	4

KERNELS$(G, 1)$:
$i = 1$: $x_i = \bar{b}$, $C = \{\bar{b}\}$, $G/\bar{b} = d + a + c$, KERNELS$(a + c + d, 2)$= $\{a + c + d\}$
$i = 2$: $x_i = d$, CUBES(G, d)= $\{\bar{b}d\}$, skip
$i = 3$: $x_i = a$, CUBES(G, a)= $\{a\bar{b}\}$, skip
$i = 4$: $x_i = c$, CUBES(G, c)= $\{\bar{b}c\}$, skip
G is not cube-free ($\bar{b}$), so G itself is not a kernel.
Ans:

$$K(G) = \{a + c + d\}$$
$$CoK(G) = \{\bar{b}\}$$

(b) Extract a multiple-cube sub-expression common to F and G.
cf: slide 11 pp. 30-31.
Let $x_c = c, x_d = d, x_{ab} = ab, x_{\bar{d}} = \bar{d}, x_{abc} = abc, x_{abd} = abd, x_{c\bar{d}} = c\bar{d}, x_a = a$.
$f_{\text{aux}} = f x_c x_d + f x_{ab} x_{\bar{d}} + f x_{abc} x_{abd} x_{c\bar{d}} + g x_a x_c x_d$
$CoK(f_{\text{aux}}) = x_c x_d$
Ans: Extract $c + d$ from F and G. Now $F = (c + d)ab + c\bar{d}, G = (c + d)\bar{b} + a\bar{b}$.