CS-472: Design Technologies for Integrated Systems

Exercise Problem Set 6 Solution — Date: 27/11/2025

Topics: Boolean operators, recursive paradigm (cf. slide set 8), symbolic and encoding optimization (cf. slide set 9)

Problem 1

Given the Boolean function $F = \bar{a}\bar{c}d + \bar{a}cd + a\bar{b}\bar{c} + abc + ac$

(a) Check if F is negative or positive unate in the variables a, b, c and d.
Ans: First, compute the cofactors:

$$
\begin{aligned}
F_a &= 0 + 0 + \bar{b}\bar{c} + bc + c = \bar{b} + c \\
F_{\bar{a}} &= \bar{c}d + cd + 0 + 0 + 0 = d \\
F_b &= \bar{a}\bar{c}d + \bar{a}cd + 0 + ac + ac = \bar{a}d + ac \\
F_{\bar{b}} &= \bar{a}\bar{c}d + \bar{a}cd + a\bar{c} + 0 + ac = a + d \\
F_c &= 0 + \bar{a}d + 0 + ab + a = a + d \\
F_{\bar{c}} &= \bar{a}d + 0 + a\bar{b} + 0 + 0 = \bar{a}d + a\bar{b} \\
F_d &= \bar{a}\bar{c} + \bar{a}c + a\bar{b}\bar{c} + abc + ac = \bar{a} + a\bar{b}\bar{c} + ac \\
F_{\bar{d}} &= 0 + 0 + a\bar{b}\bar{c} + abc + ac = a\bar{b}\bar{c} + ac
\end{aligned}
$$

- When $b = 1, c = 0, d = 1$, $F_a = 0 < F_{\bar{a}} = 1$; when $b = 0, d = 0$, $F_a = 1 > F_{\bar{a}} = 0$. Thus, F is *binate* in a.
- To have $F_{\bar{b}} = 0$, we need $a = 0, d = 0$, which implies $F_b = 0$. However, we can have $F_b = 0 < F_{\bar{b}} = 1$ when, e.g., $a = 1, c = 0$. Thus, $F_b \leq F_{\bar{b}}$ under all assignments, so F is *negative unate* in b.
- To have $F_c = 0$, we need $a = 0, d = 0$, which implies $F_{\bar{c}} = 0$. However, we can have $F_{\bar{c}} = 0 < F_c = 1$ when, e.g., $a = 1, b = 1$. Thus, $F_{\bar{c}} \leq F_c$ under all assignments, so F is *positive unate* in c.
- Observe that all minterms of $F_{\bar{d}}$ are contained by F_d, but $\bar{a}$ in F_d is not contained by $F_{\bar{d}}$. Thus, $F_{\bar{d}} \leq F_d$ and F is *positive unate* in d.

(b) Is F negative or positive unate?
Ans: F is binate because it is binate in at least one variable.

Problem 2

Given the Boolean function $G = \bar{a}\bar{b} + \bar{a}bc + \bar{a}b\bar{c}\bar{d} + ab\bar{c}d + a\bar{b} + abc$, compute:
Cofactors of G with respect to a:

$$G_a = 0 + 0 + 0 + b\bar{c}d + \bar{b} + bc = \bar{b} + c + d = \overline{(b\bar{c}\bar{d})}$$
$$G_{\bar{a}} = \bar{b} + bc + b\bar{c}\bar{d} + 0 + 0 + 0 = \bar{b} + c + \bar{d} = \overline{(b\bar{c}d)}$$

(a) The Boolean difference $\partial G/\partial a$.
Ans: $\partial G/\partial a = G_a \oplus G_{\bar{a}} = b\bar{c}$.

(b) The smoothing $S_a(G)$.
Ans: $S_a(G) = G_a + G_{\bar{a}} = 1$.

(c) The consensus $C_a(G)$.
Ans: $C_a(G) = G_a \cdot G_{\bar{a}} = \bar{b} + c$.

Problem 3

Given the Boolean function $H = \bar{a}d + ac + a\bar{b}\bar{c}$, use the positional cube notation and recursive paradigm to show if the following cubes are contained in H:

$$H = \begin{pmatrix} 10 & 11 & 11 & 01 \\ 01 & 11 & 01 & 11 \\ 01 & 10 & 10 & 11 \end{pmatrix} = \begin{pmatrix} \bar{a}d \\ ac \\ a\bar{b}\bar{c} \end{pmatrix}$$

- cd

$$cd = \begin{pmatrix} 11 & 11 & 01 & 01 \end{pmatrix}$$

$$\Rightarrow H_{cd} = \begin{pmatrix} 10 & 11 & 11 & 11 \\ 01 & 11 & 11 & 11 \\ 01 & 10 & 00 & 11 \end{pmatrix} = \begin{pmatrix} \bar{a} \\ a \\ \emptyset \end{pmatrix} = \top$$

Ans: Cube cd is contained in F.

- ad

$$ad = \begin{pmatrix} 01 & 11 & 11 & 01 \end{pmatrix}$$

$$\Rightarrow H_{ad} = \begin{pmatrix} 00 & 11 & 11 & 01 \\ 11 & 11 & 01 & 11 \\ 11 & 10 & 10 & 11 \end{pmatrix} = \begin{pmatrix} \emptyset \\ c \\ \bar{b}\bar{c} \end{pmatrix} \neq \top$$

Ans: Cube ad is not contained in F.

Problem 4

Given the constraint mapping A, find the minimum encoding matrix E that satisfies the constraints of A.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

(a) Write the dichotomies considering that the columns in A correspond to the operations AND, OR, JMP and ADD.
Ans:

$$A = \begin{array}{c} \\ \end{array} \begin{array}{cccc} AND & OR & JMP & ADD \end{array}$$

$$A = \left[\begin{array}{cccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

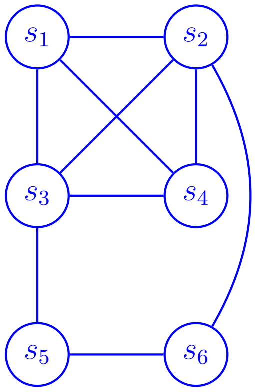
$$(\{AND, OR\}, \{JMP, ADD\})$$
$$(\{JMP, ADD\}, \{AND, OR\})$$
$$(\{OR, JMP\}, \{AND, ADD\})$$

(b) Write the seed dichotomies.
Ans:

$$(\{AND, OR\}, \{JMP\}) \rightarrow s_1$$
$$(\{AND, OR\}, \{ADD\}) \rightarrow s_2$$
$$(\{JMP, ADD\}, \{AND\}) \rightarrow s_3$$
$$(\{JMP, ADD\}, \{OR\}) \rightarrow s_4$$
$$(\{OR, JMP\}, \{AND\}) \rightarrow s_5$$
$$(\{OR, JMP\}, \{ADD\}) \rightarrow s_6$$

(c) Find the compatible seed dichotomies and draw the compatibility graph.
Ans:

$$(s_1, s_2), (s_1, s_3), (s_1, s_4),$$
$$(s_2, s_3), (s_2, s_4), (s_2, s_6),$$
$$(s_3, s_4), (s_3, s_5),$$
$$(s_5, s_6)$$



(d) Find the prime dichotomies.
Ans:

$$s_1, s_2, s_3, s_4 : (\{AND, OR\}, \{JMP, ADD\}) \quad \to p_1$$
$$s_5, s_6 : (\{OR, JMP\}, \{AND, ADD\}) \quad \to p_2$$
$$s_3, s_5 : (\{OR, JMP, ADD\}, \{AND\}) \quad \to p_3$$
$$s_2, s_6 : (\{AND, OR, JMP, \}, \{ADD\}) \quad \to p_4$$

(e) Write the covering matrix and find a minimum cover.
Ans:

	s_1	s_2	s_3	s_4	s_5	s_6
p_1	1	1	1	1	0	0
p_2	0	0	0	0	1	1
p_3	0	0	1	0	1	0
p_4	0	1	0	0	0	1

Minimum cover by selecting p_1 and p_2.

(f) Write the encoding matrix.
Ans:

$$E = \begin{array}{c} \\ AND \\ OR \\ JMP \\ ADD \end{array} \begin{array}{c} \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$