Two-level Logic Synthesis and Optimization

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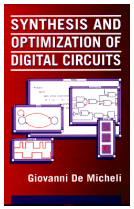


Module 1

- **Objectives**
  - **Two-level optimization**
  - **Algorithms for exact minimization**

Definitions

- Boolean variables
- Boolean literals:
  - Variables and their complement
- Product or cube:
  - Product of literals
- Implicant:
  - Product implying a value of the function (usually 1)
  - Hypercube in the Boolean space
- Minterm:
  - Product of all input variables implying a value of the function (usually 1)
  - Vertex in the Boolean space

Two-level minimization

- **Assumptions**
  - **Function represented as a Boolean cover**
    - **Set of implicants (covering all minterms)**
  - **Primary goal is to reduce the number of implicants**
  - **All implicants have the same cost**
  - **Secondary goal is to reduce the number of literals**
- **Rationale**
  - **Implicants correspond to PLA rows**
  - **Literals correspond to transistors**

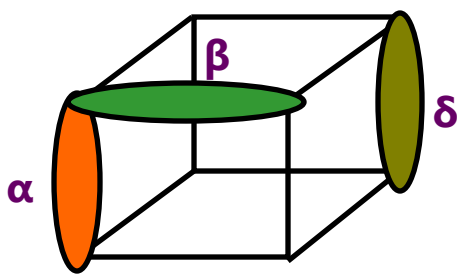
Definitions

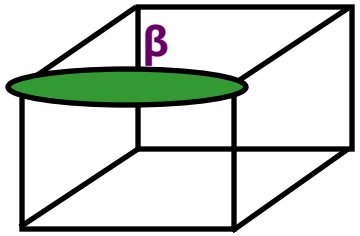
- **Minimum cover**
  - Cover of a function with minimum number of implicants
  - Global optimum
- **Minimal cover or irredundant cover**
  - Cover of the function that is not a proper superset of another cover
  - No implicant can be dropped
  - Local optimum
- **Minimal w.r.to 1-implicant containment**
  - No implicant contained by another one
  - Weak local optimum

Example

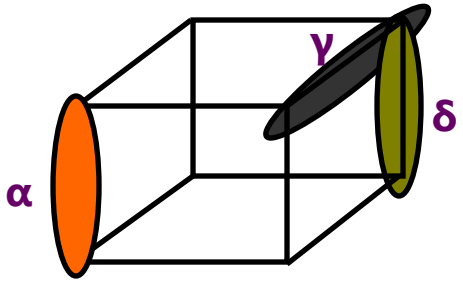
u $f_1 = a' b' c' + a' b' c + ab' c + abc + abc'$; $f_2 = a' b' c + ab' c$

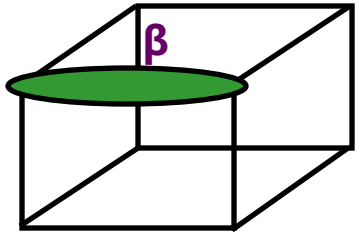
Minimum cover



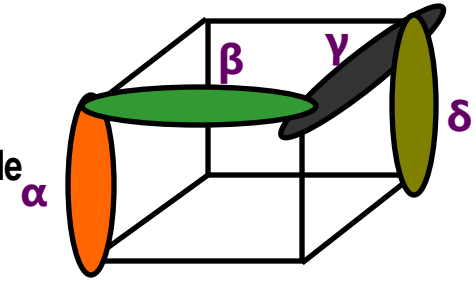


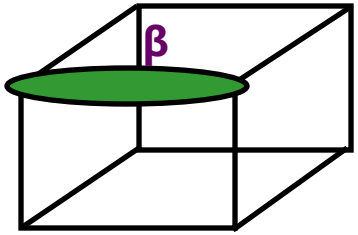
Irredundant cover





Minimal cover w.r. to single implicant containment





f1

f2

Definitions

- **Prime implicant**
  - Implicant not contained by any other implicant
- **Prime cover**
  - Cover of prime implicants
- **Essential prime implicant**
  - There exist some minterm covered only by that prime implicant
  - Needs to be included in the cover

Two-level logic minimization

- **Exact methods**
  - **Compute minimum cover**
  - **Often difficult/impossible for large functions**
  - **Based on Quine-McCluskey method**
- **Heuristic methods**
  - **Compute minimal covers (possibly minimum)**
  - **Large variety of methods and programs**
    - **MINI, PRESTO, ESPRESSO**

Exact logic minimization

- **Quine's theorem:**
  - **There is a minimum cover that is prime**
- **Consequence**
  - **Search for minimum cover can be restricted to prime implicants**
- **Quine-McCluskey method**
  - **Compute prime implicants**
  - **Determine minimum cover**

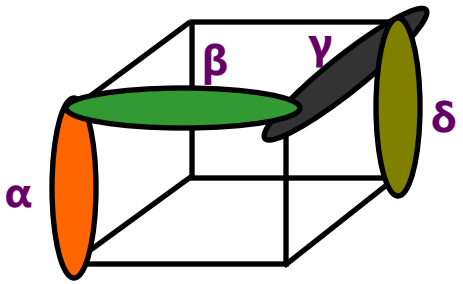
Prime implicant table

- Rows: minterms
- Columns: prime implicants
- Exponential size
  - 2^n minterms
  - Up to $3^n / n$ prime implicants
- Remarks
  - Some functions have much fewer primes
  - Minterms can be grouped together
  - Implicit methods for implicant enumeration

Example

- $f = a'b'c' + a'b'c + ab'c + abc + abc'$
- **Primes:**

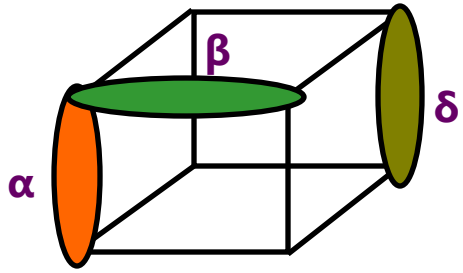
α	00*	1
β	*01	1
γ	1*1	1
δ	11*	1



Prime implicants of f

- **Table:**

	α	β	γ	δ
000	1	0	0	0
001	1	1	0	0
101	0	1	1	0
111	0	0	1	1
110	0	0	0	1



Minimum cover of f

Minimum cover
early methods

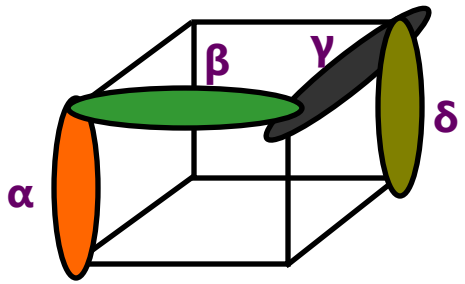
- **Reduce table**
  - **Iteratively identify essentials,**
    save them in the cover.
    Remove covered minterms
- **Petrick's method**
  - **Write covering clauses in *pos* form**
  - **Multiply out pos form into *sop* form**
  - **Select cube of minimum size**
- **Remark**
  - **Multiplying out clauses has exponential cost**

Example

- ***pos* clauses**
  - **$(\alpha)\,(\alpha+\beta)\,(\beta+\gamma)\,(\gamma+\delta)\,(\delta)=1$**
- ***sop* form:**
  - **$\alpha\beta\delta + \alpha\gamma\delta = 1$**
- **Solutions:**

$\{\,\alpha\ \beta\ \delta\,\}$

$\{\,\alpha\ \gamma\ \delta\,\}$



Matrix representation

- View table as Boolean matrix: **A**
- Selection Boolean vector for primes: **x**
- Determine **x** such that
  - $A x \geq 1$
  - Select enough columns to cover all rows
- Minimize norm (1 count) of **x**

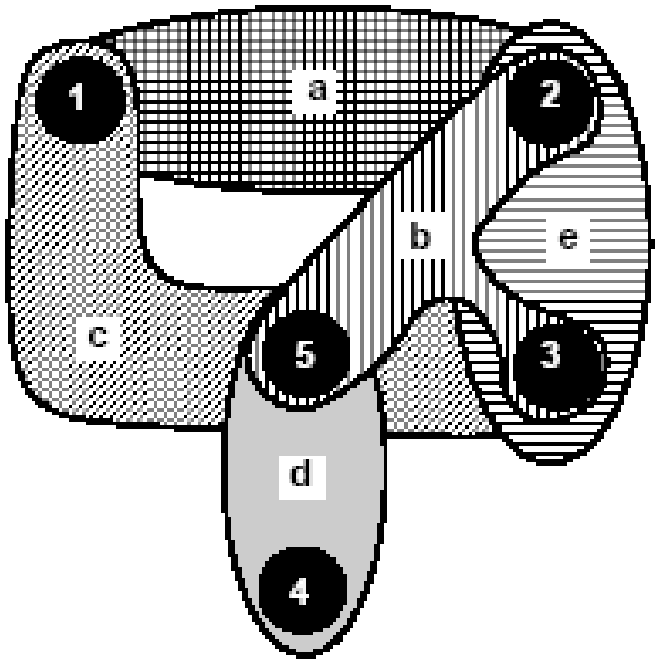
Example

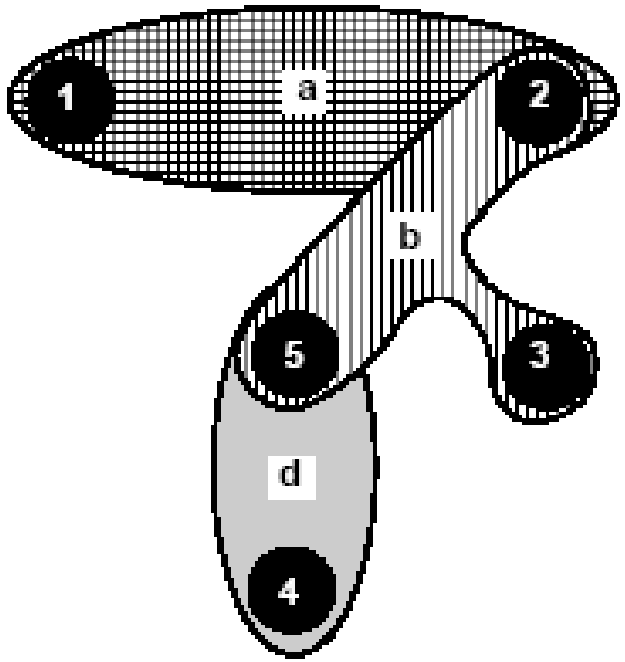
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Covering problem

- **Set covering problem:**
  - **A set S -- minterm set**
  - **A collection C of subsets (implicant set)**
  - **Select fewest elements of C to cover S**
- **Computationally intractable problem**
- **Exact solution method**
  - **Branch and bound algorithm**
- **Several heuristic approximation methods**

Example
Edge-cover of a hypergraph

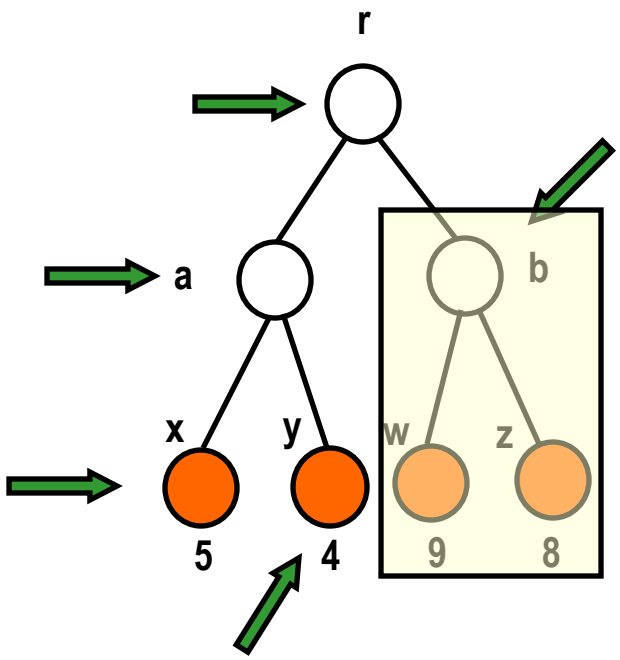




Branch and bound algorithm

- Tree search in the solution space
  - Potentially exponential
- Use bounding function:
  - If the lower bound on the solution cost
    that can be derived from a set of future choices
    exceeds the cost of the best solution seen so far,
    then kill the search
  - Bounding function should be fast to evaluate and accurate
- Good pruning may expedite the search

Example



Bound = 6

Kill sub-tree

Branch and bound for logic minimization
Reduction strategies

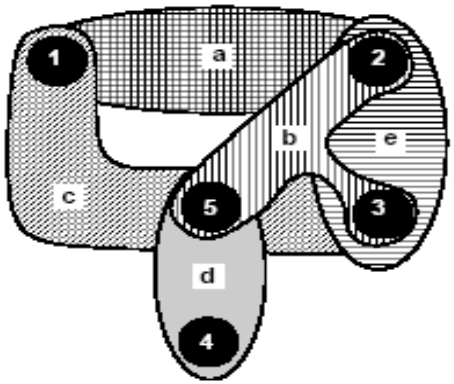
- Use matrix formulation of the problem
- Partitioning:
  - If **A** is block diagonal:
    - Solve covering problems for the corresponding blocks
- Essentials
  - Column incident to one (or more) rows with single 1
    - Select column
    - Remove covered row(s) from table

Branch and bound for logic minimization
Reduction strategies

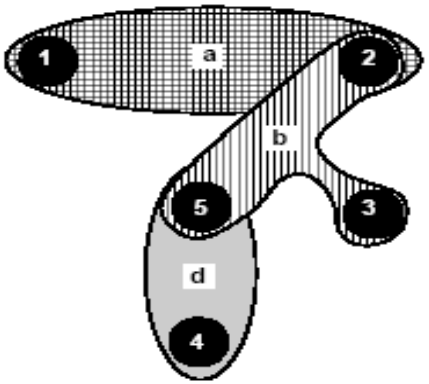
- **Column (implicant) dominance:**
  - **If $a_{ki} \geq a_{kj}$ for all k**
    - **Remove column j (dominated)**
  - **Dominated implicant (j) has its minterms already covered by dominant implicant (i)**

- **Row (minterm) dominance:**
  - **If $a_{ik} \geq a_{jk}$ for all k**
    - **Remove row i (dominant)**
  - **When an implicant covers the dominated minterm, it also covers the dominant one**

Example



(a)



(b)

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Example

- Fifth row is dominant
- Fourth column is essential
- Fifth column is dominated
- Matrix after reductions:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Branch and bound covering algorithm

```
EXACT_COVER(A,x,b) {
    Reduce matrix A and update corresponding x;
    if (current_estimate ≥ |b|) return (b);
    if (A has no rows) return(x);
    select a branching column c;
    x_c = 1;
    Ã = A after deleting c and rows incident to it;
    x˜ = EXACT_COVER(Ã,x,b);
    if ( | x˜| < |b| )
            b = x˜;
     x_c = 0;
     Ã = A after deleting c;
     x˜ = EXACT_COVER(Ã,x,b);
     if ( | x˜| < |b| )
            b = x˜;
    return(b);
}
```

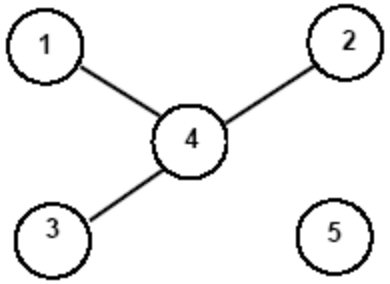
Bounding function

- Estimate lower bound on covers that can be derived from current solution vector **x**
- The sum of the 1s in **x**, plus bound of cover for local **A**
  - Independent set of rows
    - No 1 in the same column
    - Require independent implicants to cover
  - Construct graph to show pairwise independence
  - Find clique number
    - Size of the largest clique
  - Approximation (lower) is acceptable

Example

- Row 4 independent from 1,2,3
- Clique number and bound is 2

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$



Example

- There are no independent rows
  - Clique number is 1 (one vertex)
  - Bound is 1+1= 2
    - Because of the essential already selected

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Example
Branching on the cyclic core

- **Select first column**
  - Recur with $\tilde{A} = [11]$
    - Delete one dominated column
    - Take other column (essential)
  - New cost is 3
- **Exclude first column**
  - Find another solution with cost equal to 3.
  - Discard

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

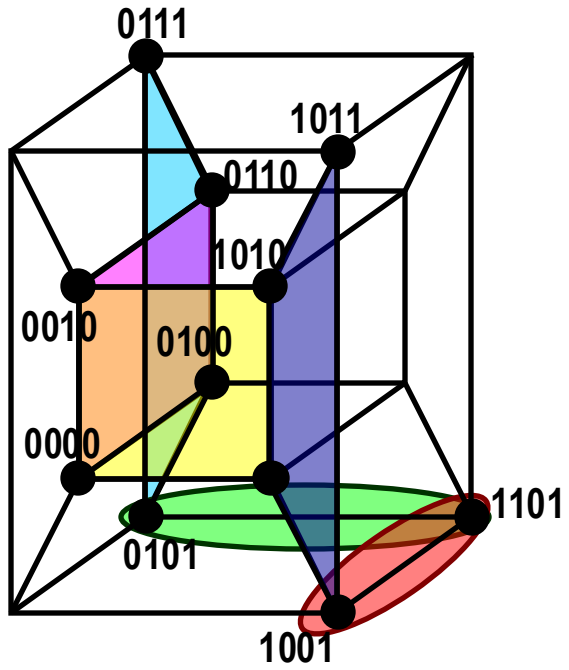
Espresso-exact

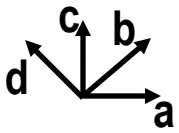
- Exact 2-level logic minimizer
- Exploits iterative reduction and branch and bound algorithm on cyclic core
- Compact implicant table
  - Rows represent groups of minterms covered by the same implicants
- Very efficient
  - Solves most benchmarks

Example

After removing the essentials

	α	β	ε	ζ
0000,0010	1	1	0	0
1101	0	0	1	1





α 0 * * 0 1

β * 0 * 0 1

γ 0 1 * * 1

δ 1 0 * * 1

ε 1 * 0 1 1

ζ * 1 0 1 1

Exact two-level minimization

- There are two main difficulties:
  - Storage of the implicant table
  - Solving the cyclic core
- Implicit representation of prime implicants
  - Methods based on binary decision diagrams
  - Avoid explicit tabulation
- Recent methods make 2-level optimization solve exactly almost all benchmarks
  - Heuristic optimization is just used to achieve solutions faster

Module 3

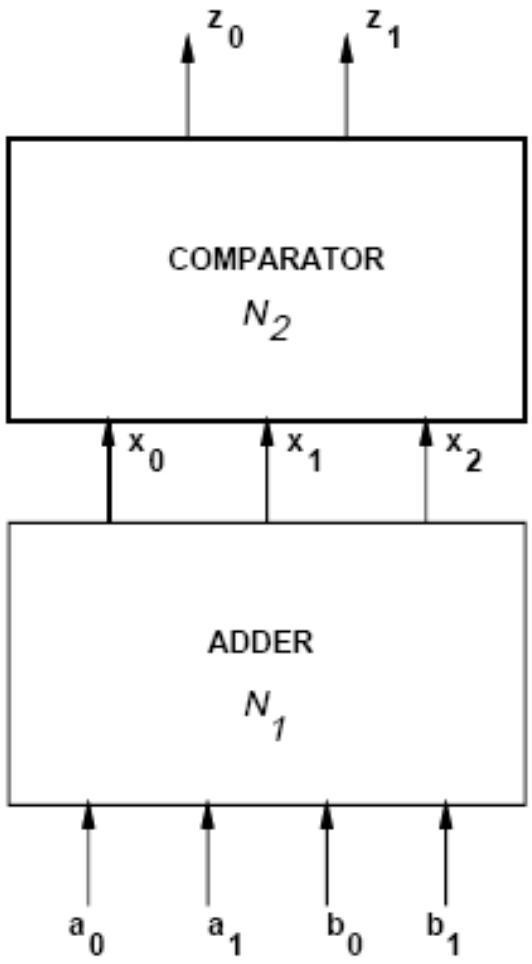
- **Boolean Relations**
  - **Motivation of using relations**
  - **Optimization of realization of Boolean relation**
  - **Comparisons to two-level optimization**

Boolean relations

- Generalization of Boolean functions
- More than one output pattern corresponds to an input pattern
  - Multiple-choice specifications
  - Model inner blocks of multi-level circuits
- Degrees of freedom in finding an implementation
  - More general than *don't care* conditions
- Problem:
  - Given a Boolean relation, find a minimum cover of a compatible Boolean function that can implement the relation

Example

- **Compare:**
  - **a + b > 4 ?**
  - **a + b < 3 ?**



Example

a_1	a_0	b_1	b_0	x
0	0	0	0	{ 000, 001, 010 }
0	0	0	1	{ 000, 001, 010 }
0	0	1	0	{ 000, 001, 010 }
0	1	0	0	{ 000, 001, 010 }
1	0	0	0	{ 000, 001, 010 }
0	1	0	1	{ 000, 001, 010 }
0	0	1	1	{ 011, 100 }
0	1	1	0	{ 011, 100 }
1	0	0	1	{ 011, 100 }
1	0	1	0	{ 011, 100 }
1	1	0	0	{ 011, 100 }
0	1	1	1	{ 011, 100 }
1	1	0	1	{ 011, 100 }
1	0	1	1	{ 101, 110, 111 }
1	1	1	0	{ 101, 110, 111 }
1	1	1	1	{ 101, 110, 111 }

Note that output 111 cannot be a+b but can be considered as a *don't care*

Example

- Circuit is no longer an adder

a_1	a_0	b_1	b_0	x
0	*	1	*	010
1	*	0	*	010
1	*	1	*	100
*	*	*	1	001
*	1	*	*	001

Minimization of Boolean relations

- Since there are many possible output values (for any input), there are many logic functions implementing the relation
  - Compatible functions
- Problem
  - Find a minimum compatible function
- Do not enumerate all compatible functions
  - Compute the primes of the compatible functions
    - C-primes
  - Derive a logic cover from the c-primes

Example

- Boolean relation:
- 4 compatible functions
- Assume c-primes include:

  ε * 1 1 01

  ζ 1 * 1 10

- To cover minterm 111:
  - Take both primes or none
  - ε ζ + ε′ ζ′
  - **Petrick's function is binate**

abc	xy
000	00
001	00
010	00
011	10
100	00
101	01
110	{00,11}
111	{00,11}

Minimizing Boolean relations

- **Minimizing Boolean *relations* is more complex**
  - **As compared to minimizing *Boolean* *functions***
- **In classic Boolean minimization we just need to select enough implicants to cover the minterms**
  - **Covering clause is unate in all variables**
  - **Any additional implicant does not hurt**
- **In Boolean relation optimization, we need to pick implicants to realize a compatible function**
  - **Some implicants cannot be taken together**
  - **Choosing an implicant may imply rejecting an other**
  - **Covering clause is binate (implicant mutual exclusion)**

Solving binate covering

- Binate cover can be solved with branch and bound
  - In practice much more difficult to solve, because it is harder to bound effectively
- Binate cover can be reduced to min-cost SAT
  - SAT solvers can be used
- Binate cover can be also modeled by BDDs
- Several approximation algorithms for binate cover

Summary: Boolean Relations

- Generalization of Boolean functions
  - More degrees of freedom than *don't care* sets
- Useful to represent multiple choices
- Useful to model internals of logic networks
- More general formalism
  - But computationally-intensive solution method