First Name: **Signature:**

Last Name:

SCIPER ID: _____

EPFL

Final exam

CS328 - Numerical Methods in Visual Computing

Procedures:

This is a timed examination. You will have a maximum of **2 hours** to complete your answers. Leave your **student ID card** on the table during the entire time so that it can be verified. Write your **first and last name** as well as your **EPFL SCIPER number** on the cover page as well as every additional sheet of paper that you request during the exam. Use only black or blue ink pens and write legibly. Exam questions done in pencil or other colored pens will **NOT** be graded. Write your solutions into the space below answers and indicate if an answer is continued elsewhere.

Multiple-choice items are each worth one point, while incorrect answers deduct one point. The final score of multiple-choice question is clamped to a non-negative number. Please try to follow Python syntax when answering questions that ask for code. Minor syntax errors are tolerated.

Regulations:

With the exception of one (1) double-sided A4 page of personal notes, any use of textbooks or other books/printed materials, formula sheets, electronic devices such as calculators, laptops, cell phones or other PDAs, MP3 players, headphones, etc. is strictly **PROHIBITED** during the examination.

A student is deemed to have failed the course if he or she is caught cheating or found to be in violation of the above regulations.

Exercise	*Max. points*	*Earned points*
1. FP Arithmetic and Interpolation	30	
2. Linear Least Squares	15	
3. The Pseudoinverse and Conditioning	25	
4. Optimization	30	
Total	**100**	

Course Name:	Numerical Methods in Visual Computing	**Date:**	27.01.2023
Course Number:	CS 328	**Time:**	15:15-17:15
Lecturer:	Wenzel Jakob		

1 FP Arithmetic and Interpolation (30 pts)

1.1 Arithmetic, Part 1 (8 pts)

The following functions repeatedly apply one of the basic arithmetic operations:

```
def add_it(x):
    for i in range(10):
        x += x
    return x
```

```
def mul_it(x):
    for i in range(10):
        x *= x
    return x
```

```
def div_it(x):
    for i in range(10):
        x /= x
    return x
```

For each of the following `print` statements, state whether the operation returns

(a) an ordinary nonzero number,
(b) ± 0,
(c) $\pm\infty$,
(d) NaN (Not a Number),
(e) or does not terminate.

You don't need to provide the actual result value. Assume that the `float32` type faithfully implements the IEEE 754 specification including handling of special values.

```
from numpy import float32 as f32

print(add_it(f32(2))) # 1
print(mul_it(f32(2))) # 2
print(div_it(f32(2))) # 3
print(div_it(f32(0))) # 4
```


1.2 Arithmetic, Part 2 (8 pts)

We now map each of the values through the function `zero_it()` provided below. Once more, classify each of the `print` statements according to the earlier list.

```
def zero_it(x):
    it = 0
    while x > 0:
        x /= 2
        it += 1
    return it

print(zero_it(add_it(f32(2)))) # 1
print(zero_it(mul_it(f32(2)))) # 2
print(zero_it(div_it(f32(2)))) # 3
print(zero_it(div_it(f32(0)))) # 4
```

1.3 Interpolation (10pts)

You are given a dataset consisting of 1000 points (x_i, y_i), where $x_i = {}^{i}/_{1000}$. Assuming that computation time does not matter, list two severe issues that you might encounter when fitting a degree-999 polynomial to this data. For each issue, explain whether this issue is avoidable or not.

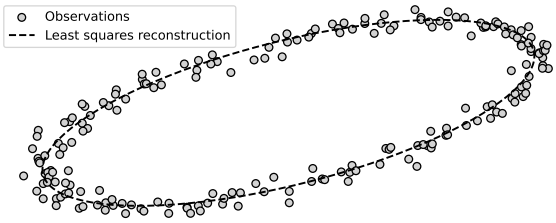
Besides addressing the above two issues, what potentially desirable property would a cubic spline interpolant have?

1.4 Multiple choice (4 pts)

True	False	
☐	☐	When subtracting two floating point values a and b that have nearly the same value, catastrophic cancellation can cause the subtraction $a - b$ to be performed extremely inaccurately.
☐	☐	Any linear combination of the first N monomials (1, x, x^2, ..) can also be expressed as a linear combination of the first N Legendre polynomials .
☐	☐	The result of point operations is always deterministic.
☐	☐	A change in the ULP (unit in the last place) for the value $3x$ is three times as large as that for x.

2 Linear Least Squares (15 pts)

You are given n data points (x_i, y_i) $(i = 1, \ldots, n)$ on an ellipse that are, however, contaminated by a significant amount of noise:



Your task is to reconstruct the parameters of this ellipse using the least squares method. An ellipse can be parameterized by constants a, b, c, d, e so that the following equation holds for positions (x, y) on the boundary:

$$ax^2 + by^2 + cxy + dx + ey - 1 = 0$$

2.1 Least Squares Fit (10 pts)

Write down the least squares system that describes this problem. You don't need to factorize the system or apply the normal equations.

2.2 Variable Noise Characteristics (5 pts)

Later, you realize that some points are much noisier than others. Luckily, the scale of the noise is predictable, and you therefore decide to perform a *weighted* least squares reconstruction that minimizes

$$E(a, b, c, d, e) = \sum_{i=1}^{N} (w_i (ax_i^2 + by_i^2 + cx_i y_i + dx_i + ey_i - 1))^2$$

Specify the least squares system for alternative problem assuming known weights w_i. As before, you don't need to factorize the system or apply the normal equations.

3 The Pseudoinverse and the Conditioning (25 pts)

In class, we had introduced the *pseudoinverse* $\mathbf{A}^+$ as a generalization of the inverse $\mathbf{A}^{-1}$. Compute the following matrix-vector products involving a pseudoinverse, and show intermediate steps or explain intuitively why the answer is correct.

Note: In expressions like $\mathbf{A}^+\mathbf{b} = \mathbf{x}$, it may be helpful to think of $\mathbf{x}$ as a particular solution of a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ that is potentially *singular*, *overdetermined*, or *underdetermined*.

Note 2: the pseudoinverse was introduced in the context of the Singular Value Decomposition (SVD), but an SVD is not needed below. You can find all answers by reasoning about the high-level properties of the pseudoinverse introduced in class.

3.1 Problem 1 (3 pts)

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}^+ \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

What is the value of x and y?

3.2 Problem 2 (7 pts)

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}^+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (x)$$

What is the value of x?

3.3 Problem 3 (7 pts)

$$\begin{pmatrix} 1/2 & 1/2 \end{pmatrix}^{+} (1) = \begin{pmatrix} x \\ y \end{pmatrix}$$

What is the value of x and y?

3.4 Condition number (4 pts)

What is the condition number of the following matrix?

$$\mathbf{A} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/4 \end{pmatrix}$$

3.5 Multiple choice (4 pts)

An ill-conditioned linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ is characterized by which of the following statements?

True	False	
☐	☐	All solutions of such a system will contain a significant amount of numerical error.
☐	☐	The singular value decomposition of $\mathbf{A}$ is not well-defined.
☐	☐	The smallest singular value of the matrix $\mathbf{A}$ is much smaller than 1.
☐	☐	A tiny relative perturbation of the right-hand side $\mathbf{b}$ can lead to a large perturbation of the solution $\mathbf{x}$.

4 Optimization (25 pts)

4.1 Backpropagation (10 pts)

In this exercise, we will compute the gradient of the function

$$\texttt{e}(\texttt{a},\texttt{b}) = \exp(\texttt{ab}) - \texttt{ab}.$$

The computer implementation of this function splits the expression into a sequence of smaller steps involving temporary values `c` and `d`.

```
a = ..        # 'a' and 'b' are
b = ..        # computed elsewhere
c = a * b
d = exp(c)
e = d - c
```

Implement a corresponding back-propagation (reverse-mode differentiation) pass that computes the derivatives `δa` and `δb` from a derivative `δe` of the output.

It may be helpful to first draw the graph structure of this computation, and annotate the sensitivity (derivative) of each node with respect to its predecessors along edges.

Note: This question is specifically about backpropagation. Other ways of computing the derivative (e.g. MATH-101 style) don't count.


```
δe = .. # δe is computed elsewhere
```

4.2 Multiple choice (5 pts)

True	False	
☐	☐	Due to the cheap gradient principle, memory usage of backpropagation will not be much bigger than that of the original program.
☐	☐	Backpropagation can compute derivatives with respect to multiple outputs of a function in an single pass.
☐	☐	Symbolic differentiation is generally more accurate than finite differences.
☐	☐	A program containing `if` statements is not differentiable and cannot be handled by automatic differentiation.
☐	☐	The main difference between automatic and MATH-101 style differentiation is the that automatc differentiation uses the chain rule.

4.3 Bisection (10 pts)

Provide an implementation of the bisection algorithm that finds a root of the function `f(x)` on the interval `[l, r]`. Your code should *not* perform *unnecessary function evaluations* in every iteration.

Implement a loop that runs for a fixed number of 10 iterations (in other words: you don't need to provide a stopping criterion). The use of complex data structures like hash tables, LRU caches, etc., is not allowed and will give 0 points.