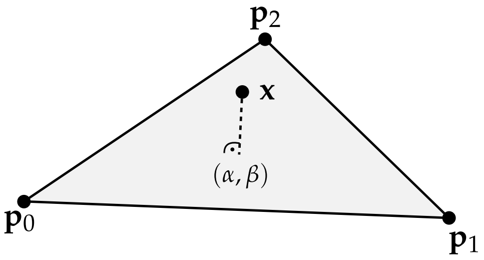
Exercise Sheet 2

Linear Least Squares

1. Triangle projection



Let the positions $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^3$ denote the corners of a triangle in three dimensions. Any point on the triangle can be expressed as a linear combination of the corner positions using local coordinates (α, β):

$$\mathbf{p}(\alpha, \beta) \coloneqq \mathbf{p}_0 + \alpha(\mathbf{p}_1 - \mathbf{p}_0) + \beta(\mathbf{p}_2 - \mathbf{p}_0)$$

Given an arbitrary point $\mathbf{x} \in \mathbb{R}^3$ that is close to the triangle, it's often useful to be able to find the values of α and β, whose associated position is *closest* to $\mathbf{x}$ according to the $\|\cdot\|_2$-norm.

(i) Re-formulate this computation as a least squares problem and express it in its standard form (i.e. $\mathbf{A}\mathbf{x} \approx \mathbf{b}$).

(ii) Derive the associated normal equations.

(iii) Suppose you have a function `Q, R = qr(A)` that produces a QR factorization of the matrix. What precise sequence of steps is needed to obtain $\mathbf{x}$ using this factorization?

Solution:

(i) Let

$$\mathbf{A} = \begin{pmatrix} | & | \\ \mathbf{p}_1 - \mathbf{p}_0 & \mathbf{p}_2 - \mathbf{p}_0 \\ | & | \end{pmatrix}, \quad \mathbf{b} = \mathbf{x} - \mathbf{p}_0.$$

Then the least squares problem is in standard form

$$\mathbf{A}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \approx \mathbf{b}.$$

(ii) Normal equations $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$:

$$\begin{pmatrix} \|\mathbf{p}_1 - \mathbf{p}_0\|^2 & \langle \mathbf{p}_1 - \mathbf{p}_0, \mathbf{p}_2 - \mathbf{p}_0 \rangle \\ \langle \mathbf{p}_2 - \mathbf{p}_0, \mathbf{p}_1 - \mathbf{p}_0 \rangle & \|\mathbf{p}_2 - \mathbf{p}_0\|^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \langle \mathbf{p}_1 - \mathbf{p}_0, \mathbf{x} - \mathbf{p}_0 \rangle \\ \langle \mathbf{p}_2 - \mathbf{p}_0, \mathbf{x} - \mathbf{p}_0 \rangle \end{pmatrix}$$

(iii) Using QR factorization where $\mathbf{A} = \mathbf{Q}\mathbf{R}$:

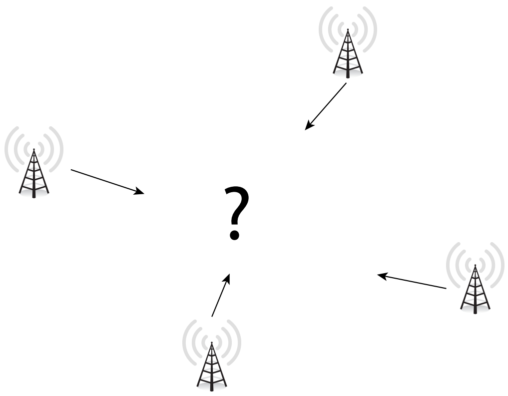
- a) Compute $\mathbf{Q}, \mathbf{R} = \mathtt{qr}(\mathbf{A})$ where $\mathbf{A}$ is the 3×2 matrix from part (i)
- b) Compute $\mathbf{c} = \mathbf{Q}^T\mathbf{b}$ (matrix-vector multiplication)

c) Solve $\mathbf{Rx} = \mathbf{c}$ by back substitution (since $\mathbf{R}$ is upper triangular)

Note: Since $\mathbf{A}$ is 3×2, we only need the first 2 columns of $\mathbf{Q}$ and the 2×2 upper-triangular part of $\mathbf{R}$.

2. Radar localization

A signal source is briefly observed by n radar dishes with *distinct* positions (x_i, y_i) $(i = 1, \ldots, n)$ scattered on a 2D plane. Following the observation, the operator of each dish reports back a unit vector (u_i, v_i) representing the most likely direction towards the signal.



We'll now focus on the problem of computing the source position (s, t) given these observations.

(i) Suppose that $n = 3$. For what dish arrangement can we expect the associated numerical problem to be ill-conditioned? Does the problem still admit solutions in this case? Justify your answers.

(ii) Let's assume from now on that the radar dish locations were intelligently chosen to avoid the problems discussed in the previous question.

For what values of n is the problem always underdetermined or overdetermined regardless of the source position? Why?

(iii) Let's assume that there are only two dishes. Construct a square linear system that can be used to solve for (s, t).

Solution:

(i) If all radar stations are arranged on a line, they will not be able to localize sources on or very close to the line, hence the problem is ill-conditioned. However, sources at some distance can still be triangulated without problems.

(ii) Each station leads to a constraint that requires the source to lie on a line. The problem is always underdetermined for $n = 1$, since a single line is not enough to determine a position. The problem is always overdetermined for $n > 3$. In two dimensions, it depends on how the line constraint is specified. A parametric form leads to two equations per line, i.e. an overdetermined system, while implicit form (one equation per line) produces a square system given that we are working in a 2D coordinate system.

(iii) The observation of each radar dish specifies that the signal source is located on a line. For dish i, this gives the parametric equation:

$$(s, t) = (x_i, y_i) + \alpha_i(u_i, v_i) \quad \text{for some } \alpha_i \in \mathbb{R}.$$

Approach 1 (Parametric form, square system): Keep the unknown along-line distances α_i, which measure how far we move from dish i in direction (u_i, v_i) to reach the source. For dishes $i = 1, 2$ the two coordinates give

$$s = x_i + \alpha_i u_i, \qquad t = y_i + \alpha_i v_i.$$

Stacking these yields a square 4×4 linear system for the unknowns $(s, t, \alpha_1, \alpha_2)^T$:

$$\begin{pmatrix} 1 & 0 & -u_1 & 0 \\ 0 & 1 & -v_1 & 0 \\ 1 & 0 & 0 & -u_2 \\ 0 & 1 & 0 & -v_2 \end{pmatrix} \begin{pmatrix} s \\ t \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}.$$

This problem can be solved directly, e.g., by Gaussian elimination. More observations would lead to least squares solution.

Approach 2 (Implicit form): Convert each parametric line to implicit form. A line through (x_i, y_i) with direction (u_i, v_i) can be written using a perpendicular normal vector $\mathbf{n}_i = (-v_i, u_i)^T$:

$$\mathbf{n}_i \cdot \begin{pmatrix} s - x_i \\ t - y_i \end{pmatrix} = 0$$

Expanding this for dish i:

$$-v_i(s - x_i) + u_i(t - y_i) = 0 \quad \Rightarrow \quad -v_i s + u_i t = -v_i x_i + u_i y_i$$

Stacking equations for dishes 1 and 2 gives the 2×2 linear system:

$$\begin{pmatrix} -v_1 & u_1 \\ -v_2 & u_2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} -v_1 x_1 + u_1 y_1 \\ -v_2 x_2 + u_2 y_2 \end{pmatrix}$$