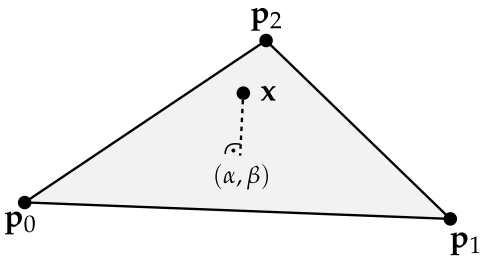
Exercise Sheet 2

Linear Least Squares

1. Triangle projection



Let the positions $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^3$ denote the corners of a triangle in three dimensions. Any point on the triangle can be expressed as a linear combination of the corner positions using local coordinates (α, β):

$$\mathbf{p}(\alpha, \beta) \coloneqq \mathbf{p}_0 + \alpha(\mathbf{p}_1 - \mathbf{p}_0) + \beta(\mathbf{p}_2 - \mathbf{p}_0)$$

Given an arbitrary point $\mathbf{x} \in \mathbb{R}^3$ that is close to the triangle, it's often useful to be able to find the values of α and β, whose associated position is *closest* to $\mathbf{x}$ according to the $\|\cdot\|_2$-norm.

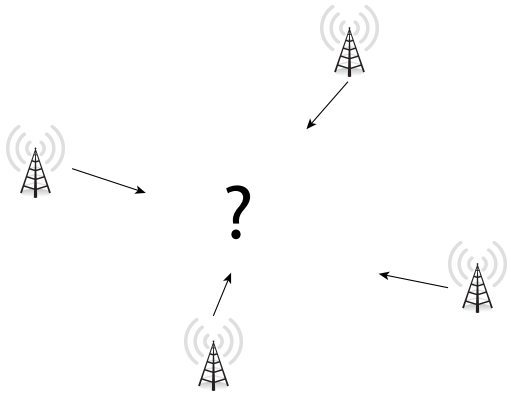
(i) Re-formulate this computation as a least squares problem and express it in its standard form (i.e. $\mathbf{Ax} \approx \mathbf{b}$).

(ii) Derive the associated normal equations.

(iii) Suppose you have a function `Q, R = qr(A)` that produces a QR factorization of the matrix. What precise sequence of steps is needed to obtain $\mathbf{x}$ using this factorization?

2. Radar localization

A signal source is briefly observed by n radar dishes with *distinct* positions (x_i, y_i) $(i = 1, \ldots, n)$ scattered on a 2D plane. Following the observation, the operator of each dish reports back a unit vector (u_i, v_i) representing the most likely direction towards the signal.



We'll now focus on the problem of computing the source position (s, t) given these observations.

(i) Suppose that $n = 3$. For what dish arrangement can we expect the associated numerical problem to be ill-conditioned? Does the problem still admit solutions in this case? Justify your answers.

(ii) Let's assume from now on that the radar dish locations were intelligently chosen to avoid the problems discussed in the previous question.

For what values of n is the problem always underdetermined or overdetermined regardless of the source position? Why?

(iii) Let's assume that there are only two dishes. Construct a square linear system that can be used to solve for (s, t).