Principles of Online Decision-Making (CS-303)

Problem Set 4 - Solutions

Problem 1

(a) Let X be a Markov chain. Which of the following are also Markov chains?

1. X_{m+r} for $r \geq 0$.
2. X_{2m} for $m \geq 0$.
3. The sequence of pairs (X_n, X_{n+1}) for $n \geq 0$.

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1. This is a MC! Using the Markov property,

$$\mathbb{P}[X_{m+r} = k | X_m = i_m, \ldots, X_{m+r-1} = i_{m+r-1}] = \mathbb{P}[X_{m+r} = k | X_{m+r=1} = i_{m+r-1}]. \quad (1)$$

2. This is a MC! Let $A = \{X_{2r} = i_{2r} \text{ for } 0 \leq r \leq m\}$ (all the even indices up to $2m$) and $B = \{X_{2r+1} = i_{2r+1} \text{ for } 0 \leq r \leq m-1\}$ (all the odd indices up to $2m-1$). Then

$$\begin{aligned}
\mathbb{P}[X_{2m+2} = k | A] &= \sum_{i_1, i_3, \ldots, i_{2m+1}} \frac{\mathbb{P}[X_{2m+2} = k, X_{2m+1} = i_{2m+1}, A, B]}{\mathbb{P}[A]} && (2)\\
&= \sum_{i_1, i_3, \ldots, i_{2m+1}} \frac{\mathbb{P}[X_{2m+2} = k, X_{2m+1} = i_{2m+1} | A, B]\mathbb{P}[A, B]}{\mathbb{P}[A]} && (3)\\
&= \sum_{i_1, i_3, \ldots, i_{2m+1}} \frac{\mathbb{P}[X_{2m+2} = k, X_{2m+1} = i_{2m+1} | X_{2m} = i_{2m}]\mathbb{P}[A, B]}{\mathbb{P}[A]} && (4)\\
&= \mathbb{P}[X_{2m+2=k} | X_{2m} = i_{2m}], && (5)
\end{aligned}$$

because

$$\sum_{i_1, i_3, \ldots, i_{2m-1}} \mathbb{P}[B|A] = 1 \quad (6)$$

and

$$\sum_{i_{2m+1}} \mathbb{P}[X_{2m+2} = k, X_{2m+1} = i_{2m+1} | X_{2m} = i_{2m}] = \mathbb{P}[X_{2m+2=k} | X_{2m} = i_{2m}]. \quad (7)$$

3. This is a MC! With $Y_n = (X_n, X_{n+1})$,

$$\begin{aligned}
\mathbb{P}[Y_{n+1} = (k,l) | Y_0 = (i_0, i_1), \ldots, Y_n = (i_n, k)] &= \mathbb{P}[Y_{n+1} = (k,l) | X_{n+1} = k] && (8)\\
&= \mathbb{P}[Y_{n+1} = (k,l) | Y_n = (i_n, k)], && (9)
\end{aligned}$$

by the Markov property of X.

(b) Let X be a Markov chain. Show that for $1 < r < n$,

$$\mathbb{P}[X_r = k | X_i = x_i \text{ for } i = 1, 2, \ldots, r-1, r+1, \ldots, n] = \mathbb{P}[X_r = k | X_{r-1} = x_{r=1}, X_{r+1} = x_{r+1}] \quad (10)$$

$$
\begin{aligned}
\mathbb{P}[X_r = k|X_i = x_i \text{ for } i = 1,2,\dots,r-1,r+1,\dots,n] &= \frac{\mathbb{P}[X_1 = x_1]p_{x_1,x_2}\dots p_{x_{r-1},k}p_{k,x_{r+1}}\dots p_{x_{n-1},x_n}}{\mathbb{P}[X_1 = x_1]\dots p_{x_{r-1},x_{r+1}}(2)\dots p_{x_{n-1},x_n}} && (11)\\
&= \frac{\mathbb{P}[X_1 = x_1]p_{x_{r-1},k}p_{k,x_{r+1}}}{\mathbb{P}[X_1 = x_1]p_{x_{r-1},x_{r+1}}(2)\dots p_{x_{n-1},x_n}} && (12)\\
&= \mathbb{P}[X_r = k|X_{r-1} = x_{r=1}, X_{r+1} = x_{r+1}] && (13)
\end{aligned}
$$

This is the two-sided Markov property: given the two "neighbors" of X_r, X_r is conditionally independent of the entire past and the entire future of the process.

Problem 2

(a) Show that a state i is recurrent iff the expected number of visits to state i, having started at i, is infinite.

Let $\mathbb{I}_k$ be the indicator function of the event $\{X_k = i\}$. Then the expected number of visits to state i is $N = \sum_{k=0}^{\infty} \mathbb{I}_k$. Then

$$\mathbb{E}[N] = \sum_{k=0}^{\infty} \mathbb{E}[\mathbb{I}_k] = \sum_{k=0}^{\infty} p_{ii}(k). \quad (14)$$

(b) Let X be a Markov chain containing an absorbing state s with which all other states communicate (i.e., $p_{is}(n) > 0$ for some $n = n(i)$). Show that all states other than s are transient.

Let $i \neq s$ be a state, and define $n_i = \min\{n : p_{is} > 0\}$. If $X_0 = i$ and $X_{n_i} = s$, then X cannot make a visit to i in the interval $[1, n_i - 1]$, given the definition of n_i. By assumption, s is absorbing, which implies

$$\mathbb{P}[\text{no return to } i|X_0 = i] \geq \mathbb{P}[X_{n_i} = s|X_0 = i] > 0, \quad (15)$$

hence i is transient.

Problem 3

We have a Markov chain with the following transition matrix:

$$P = \begin{pmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{pmatrix} \quad (16)$$

Compute $p_{ij}(n)$ and the mean recurrence time of all states.

If $p = 0$, all states are absorbing.

If $p > 0$, we diagonalize $P = B\Lambda B^{-1}$ with

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad (17)$$

$$B^{-1} = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & -1/2 \\ 1/4 & -1/2 & 1/4 \end{pmatrix}, \tag{18}$$

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-2p & 0 \\ 0 & 0 & 1-4p \end{pmatrix}. \tag{19}$$

Therefore,

$$P^n = B\Lambda^n B^{-1} = B \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1-2p)^n & 0 \\ 0 & 0 & (1-4p)^n \end{pmatrix} B^{-1}, \tag{20}$$

from which $p_{ij}(n)$ can be determined:

$$p_{11}(n) = p_{33}(n) = \frac{1}{4} + \frac{1}{2}(1-2p)^n + \frac{1}{4}(1-4p)^n, p_{22}(n) = \frac{1}{2} + \frac{1}{4}(1-4p)^n. \tag{21}$$

For the mean recurrence time, we start with $F_{ii}(s) = 1 - P_{ii}(s)^{-1}$, where

$$P_{11}(s) = P_{33}(s) = \frac{1}{4(1-s)} + \frac{1}{2(1-s(1-2p))} + \frac{1}{4(1-s(1-4p))} \tag{22}$$

$$P_{22}(s) = \frac{1}{2(1-s)} + \frac{1}{2(1-s(1-4p))}. \tag{23}$$

From $\mu_i = F'_{ii}(1)$, we can compute $\mu_1 = \mu_3 = 4$ and $\mu_2 = 2$.