Problem 1. *(LDPC-Decoder)* (Paper and Pencil)

(a)

$$H = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}.$$

(b)

$$S = \begin{pmatrix} 1 & 4 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}.$$

(c) To simplify the notation, we write V_{ij} instead of $V_{i\to j}$ and similarly for F_{ij}. Each message from a check node is computed as:

$$F_{mn}(x_n) = \sum_{\sim x_n} f_m(\boldsymbol{x}) \prod_{n' \in \mathcal{F}(m)\backslash n} V_{n'm}(x_{n'}).$$

So, in our case:

$$\begin{aligned} F_{43}(x_3) &= \sum_{\sim x_3} f_4(x_1, x_2, x_3) \prod_{n' \in \mathcal{F}(4)\backslash 3} V_{n'4}(x_{n'}) \\ &= \sum_{x_1, x_2} f_4(x_1, x_2, x_3) V_{14}(x_1) V_{24}(x_2) \\ &= \sum_{x_1, x_2} f_4(x_1, x_2, x_3) f_1(x_1) f_2(x_2), \end{aligned}$$

where we used the fact that the messages from the variable nodes are $V_{14}(x_1) = f_1(x_1)$ and $V_{24}(x_2) = f_2(x_2)$. Next, we compute

$$F_{43}(2) = \sum_{x_1, x_2} f_4(x_1, x_2, 2) f_1(x_1) f_2(x_2).$$

There are $3^2 = 9$ terms in the summation above. However, we notice that all the terms with $x_1 = 0$ as well as the ones with $x_2 = 2$ are zero, since $f_1(0) = 0$ and $f_2(2) = 0$. We are left with 4 combinations of (x_1, x_2): $(1,0), (1,1), (2,0), (2,1)$. From all these combinations, only $(1,0)$ will lead to a non-vanishing value of $f_4(x_1, x_2, 2)$. So in the end we obtain $F_{43}(2) = f_1(1) f_2(0) = 2 \cdot 1 = 2$.