SOLUTION 1.

1. Let us compute the noiseless channel output:

$$\begin{aligned} R(t) - N(t) &= s(t) \star h(t) \\ &= \left(\sum_k s_k \psi(t - kT) \right) \star h(t) \\ &= \sum_k s_k \psi(t - kT) \star h(t) \\ &= \sum_k s_k \phi(t - kT). \end{aligned}$$

   This shows $R(t) - N(t) = \tilde{s}(t)$ which is precisely equivalent to a system communicating data via the pulse shape $\phi(t)$ through an ideal AWGN channel.

2. We use linearity. For the signal (as opposed to noise) contribution, the input is a sum of terms of the form $s_j \phi(t - jT)$. The contribution of such a term is

$$\langle s_j \phi(t - jT), \psi(t - kT) \rangle = s_j \langle \phi(t - jT), \psi(t - kT) \rangle = s_j \langle \phi(t + (k - j)T), \psi(t) \rangle,$$

   which can be written as $s_j l_{k-j}$ with

$$l_i = \langle \phi(t + iT), \psi(t) \rangle = \rho_{\phi,\psi}(iT).$$

   The contribution of the noise is $Z_k = \langle N(t), \psi(t - kT) \rangle$. From PDC, we know that $Z_k \sim \mathcal{N}(0, N_0/2)$ if $N(t)$ is real-valued and $Z_k \sim \mathcal{N}_{\mathcal{C}}(0, N_0)$ if $N(t)$ is complex-valued.

   Hence

$$Y_k = \sum_j s_j l_{k-j} + Z_k = \sum_i s_{k-i} l_i + Z_k = s_k l_0 + \sum_{i \neq 0} l_i s_{k-i} + Z_k.$$

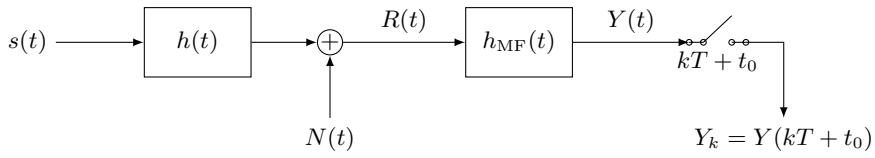
   If ψ is a Nyquist pulse with bandwidth B and $h(t)$ is an ideal low-pass filter with bandwidth at least B, then $\phi(t) = \psi(t) \star h(t) = \psi(t)$. Consequently, since ψ is a Nyquist pulse,

$$l_j = \rho_{\psi,\psi}(jT) = \int \psi(\alpha + jT) \psi^*(\alpha) \, d\alpha = \mathbb{1}\{j = 0\}.$$

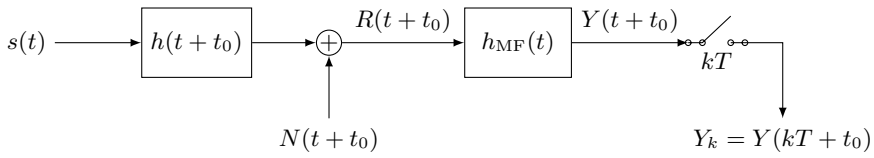
3. Because $\psi(t - iT)$ and $\psi(t - jT)$ are orthogonal when $i \neq j$, we know (PDC) that Z_i and Z_j are independent. Hence $\{Z_k\}$ is i.i.d.

SOLUTION 2.

1. Suppose that the channel impulse response is $h(t)$, that $R(t)$ is the input to the matched filter of impulse response $h_{MF}(t)$ and $Y(t)$ is the MF output (next figure).



   Linearity implies that if we anticipate the channel impulse response (i.e. we substitute $h(t)$ with $h(t + t_0)$), then all the signals are anticipated. In particular, the MF output is $Y(t + t_0)$ (next figure). If we sample it at $t = kT$, we obtain the same as sampling $Y(t)$ at $t = kT + t_0$.



2. In the definition of l_j, we substitute jT with $jT + t_0$. Since the channel is transparent, $\phi(t) = \psi(t)$. Hence

$$l_j = \rho_{\psi,\psi}(jT + t_0) = \int \psi(\alpha + jT + t_0)\psi^*(\alpha)\ d\alpha.$$

3. (a) The convolution of a sinc with itself is again a sinc. (This is easily seen in the frequency domain.) Hence

$$\rho(t) = \int \psi(\alpha + t)\psi^*(\alpha)\ d\alpha = \mathrm{sinc}(t)$$

when $\psi(t) = \mathrm{sinc}(t)$. Therefore

$$l_j = \mathrm{sinc}(j + t_0) = \frac{\sin(j\pi + t_0\pi)}{j\pi + t_0\pi} = (-1)^j \frac{\sin(\pi t_0)}{j\pi + t_0\pi}$$

(b) The ISI term equals

$$\sum_{j\neq 0} s_{k-j}(-1)^j \frac{\sin(\pi t_0)}{\pi j + \pi t_0}.$$

If the data symbols are such that $s_{k-j}(-1)^j$ is positive for all $j \neq 0$ (or negative for all $j \neq 0$), the above summation diverges and ISI will be unbounded.

Solution 3.

1. See the provided MATLAB/Python routine. The diagram shows the presence of ISI in the system; if we look at the 'eye' at time $t = 0$ we see that not all traces of the signal pass through the points ± 1. In other words, the height of the eye is not 2 (which is the distance between two data symbols ± 1). If the pulse had infinite duration, since the noise is zero, the traces would have all passed through ± 1 at time $t = 0$.

2. See the provided MATLAB/Python script. The result will be something like the plots in Figures 1–3. We see that for $\beta = 0$ (the sinc pulse) with both truncation lengths we have ISI. For $\beta = 0.3$ we have an ISI-free system if we truncate the pulse to $20T$. Finally with $\beta = 0.9$, we never have ISI in the system. Note that the price we pay for avoiding the ISI is bandwidth. Also note that for all values of β, if the pulse would have had infinite duration, since the noise is zero, the traces would have all passed through ± 1 at time $t = 0$ (no ISI).

Solution 4.

See the provided MATLAB/Python script.

1. The output for different signal-to-noise ratios looks similar to what you see in Figures 4–6. It can be seen that as the signal-to-noise ratio increases the constellation clouds shrink and the eye becomes more open. Indeed, the openness of the eye is directly related to the quality of transmission.

2. Channel delay, as we expected, 'closes' the eye (and expands the constellation clouds) as it can be seen in Figures 7 and 8.

3. The pulse with roll-off factor $\beta = 0.9$ is more resistant against delays. We see in Figures 9–11 that for a given amount of delay, the eyes for the pulse with $\beta = 0.9$ are larger.

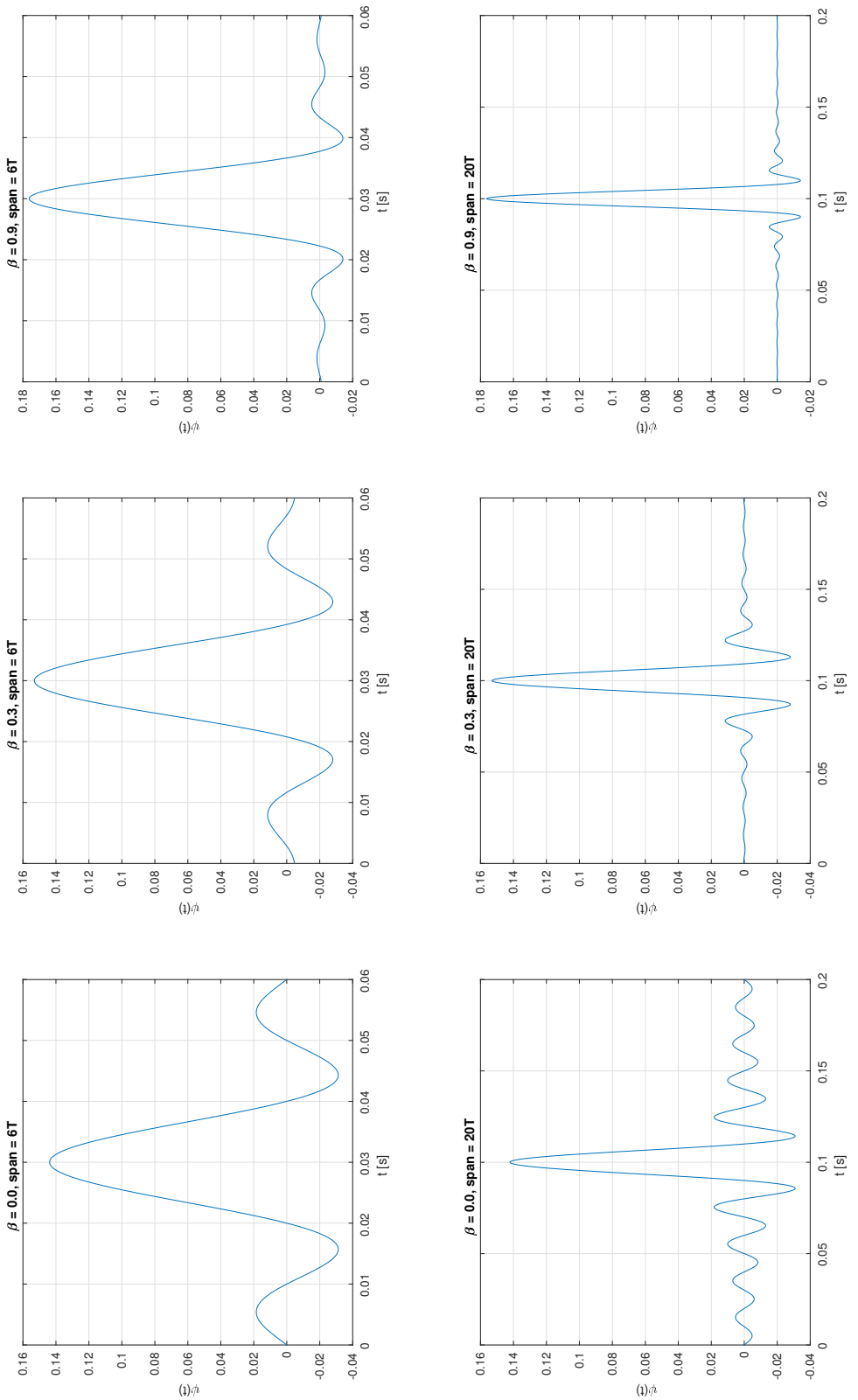


Figure 1: The time-domain root-raised-cosine pulses.

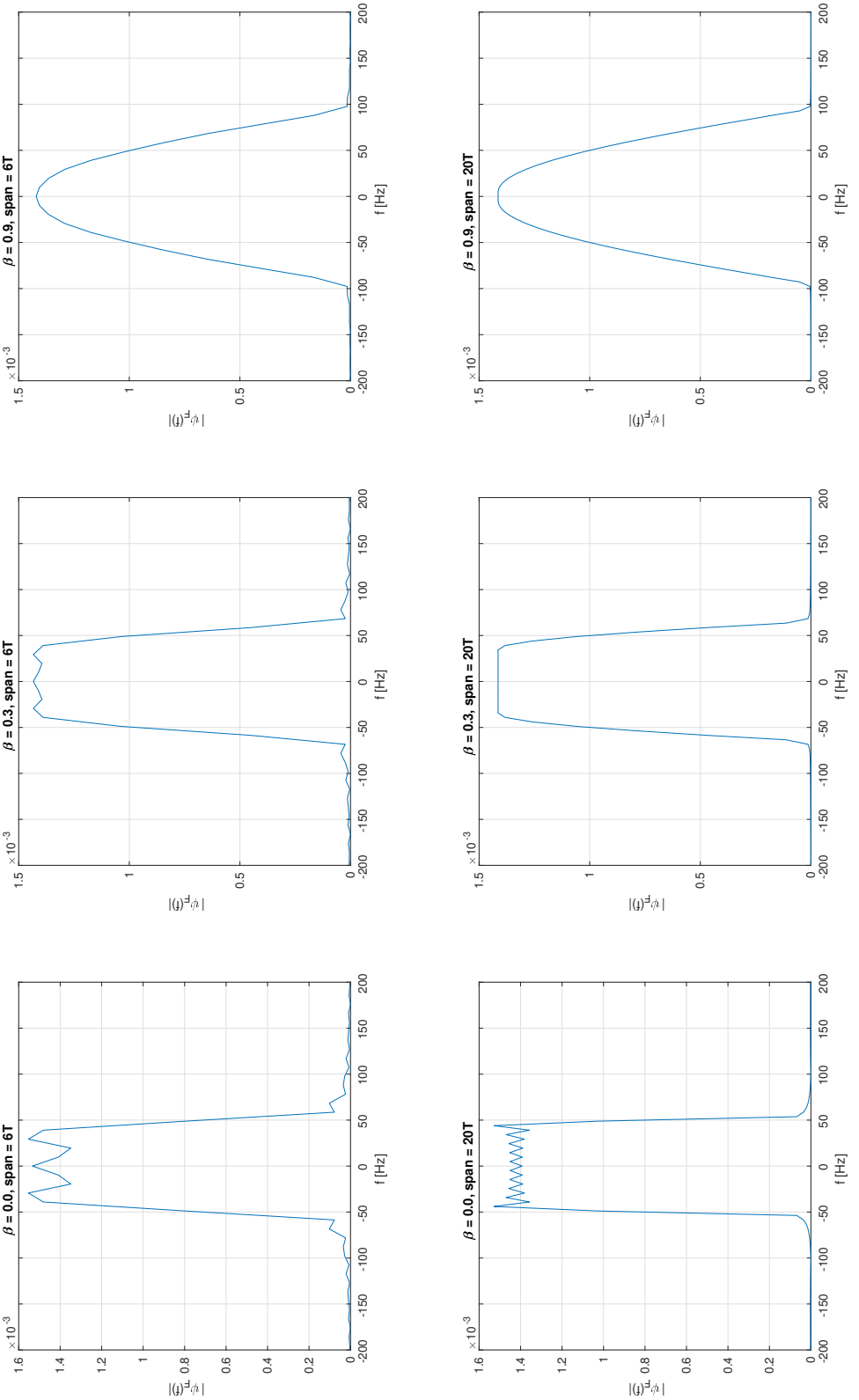


Figure 2: The spectra of root-raised-cosine pulses.

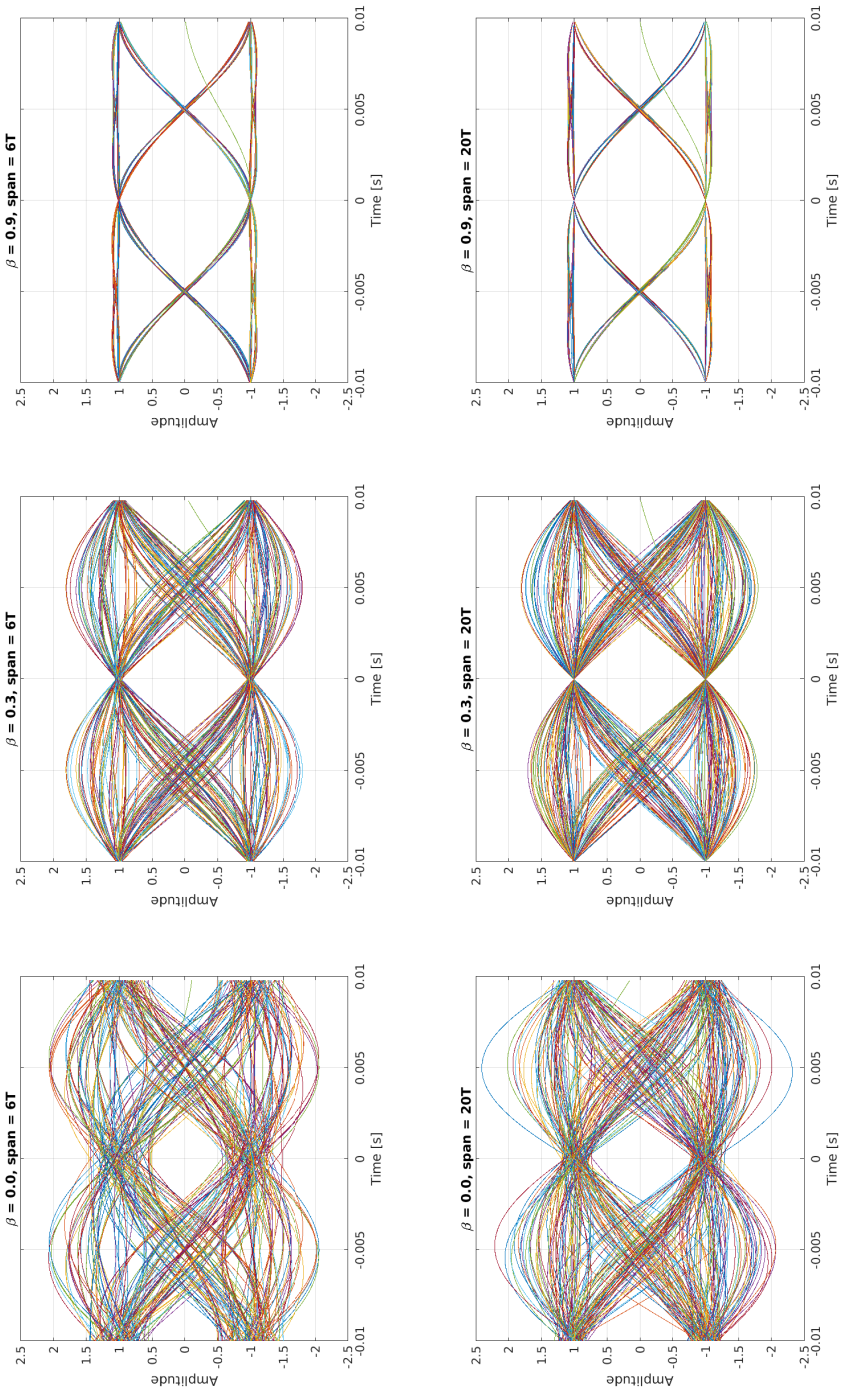


Figure 3: The eye diagrams of root-raised-cosine pulses.

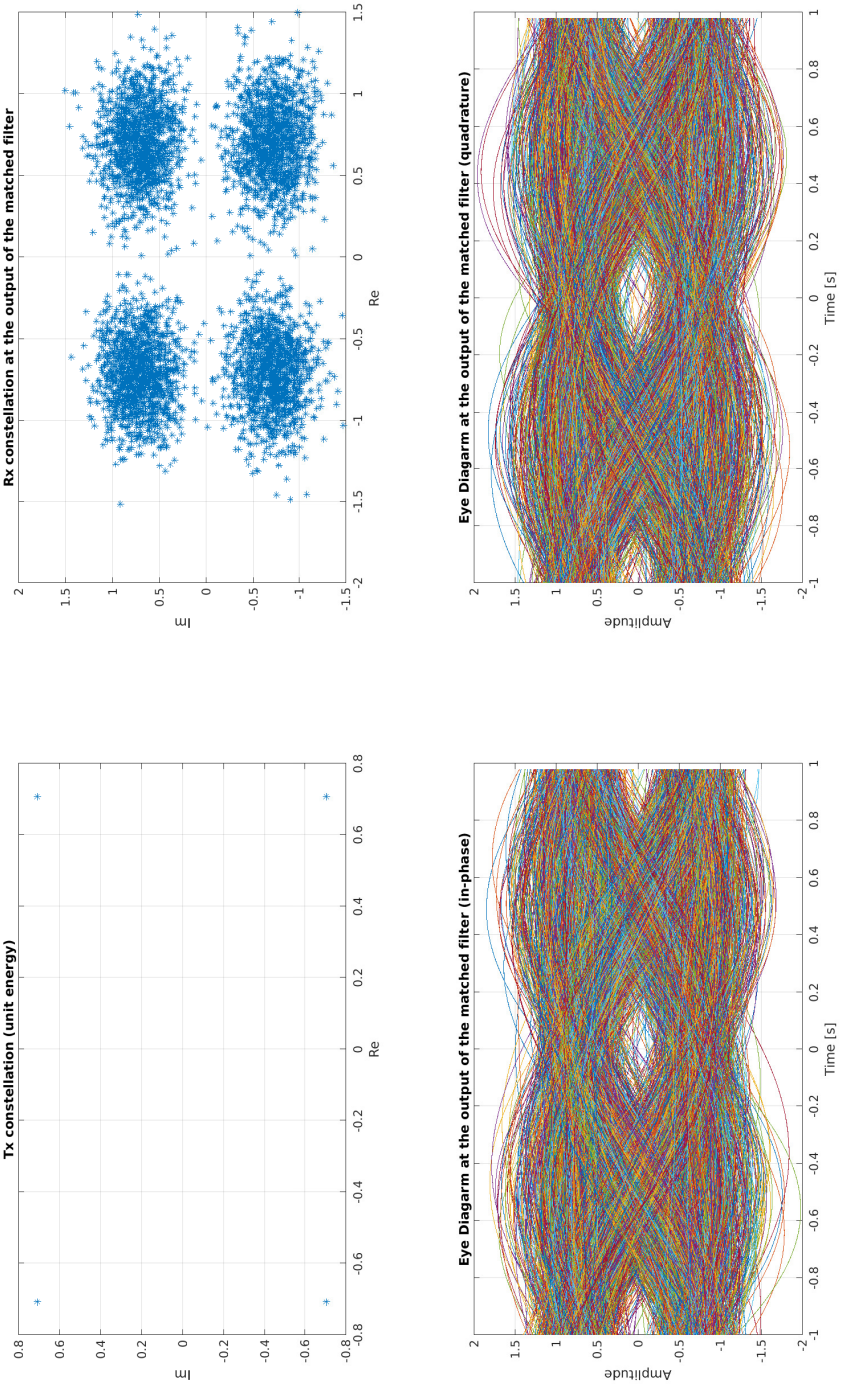


Figure 4: Constellation Cloud and Eye Diagram for $E_s/N_0 = 10$ dB

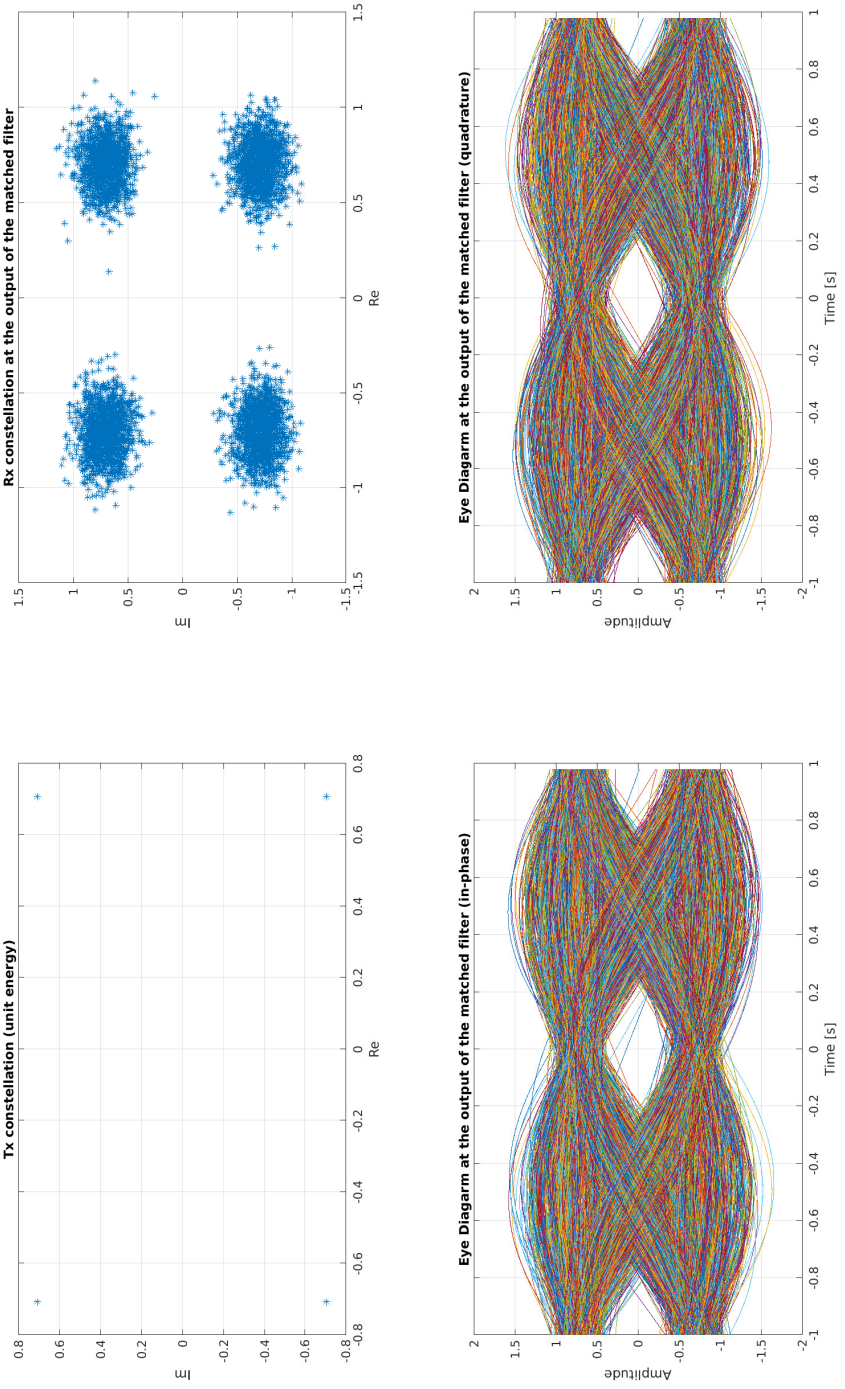


Figure 5: Constellation Cloud and Eye Diagram for $E_s/N_0 = 15$ dB

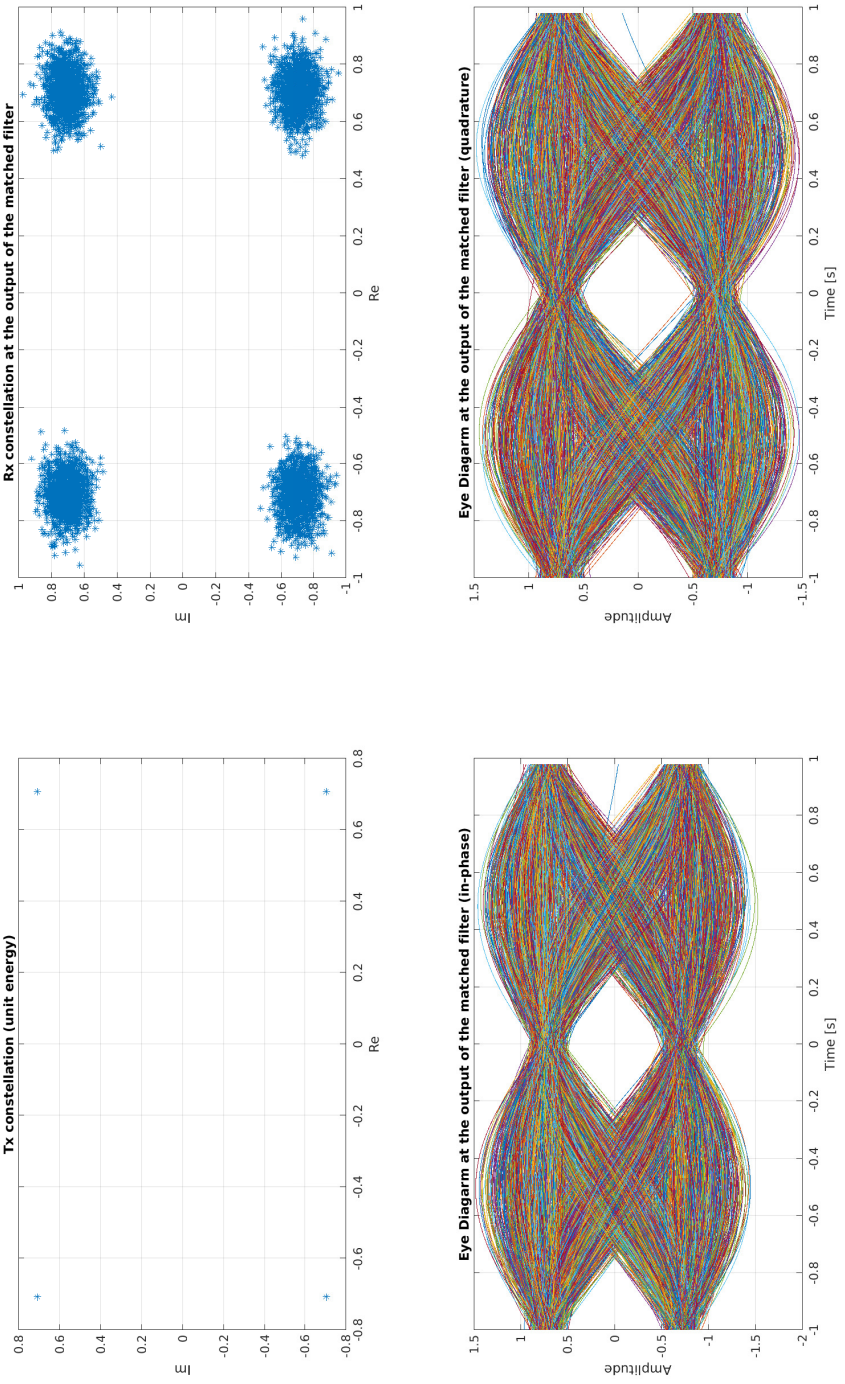


Figure 6: Constellation Cloud and Eye Diagram for $E_s/N_0 = 20$ dB

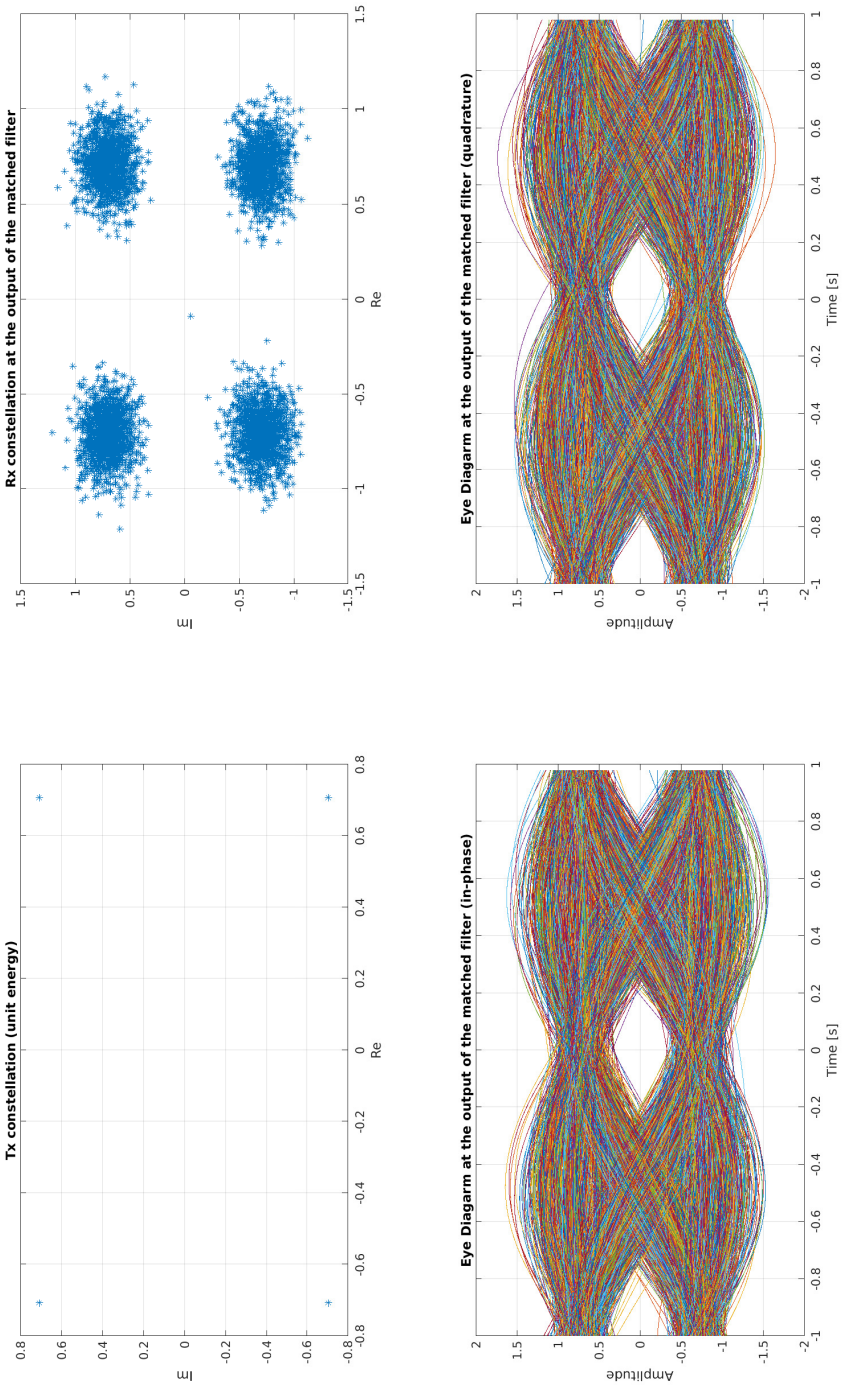


Figure 7: Constellation cloud and eye diagram for $E_s/N_0 = 15$ dB when channel introduces $d = 1$ sample of delay

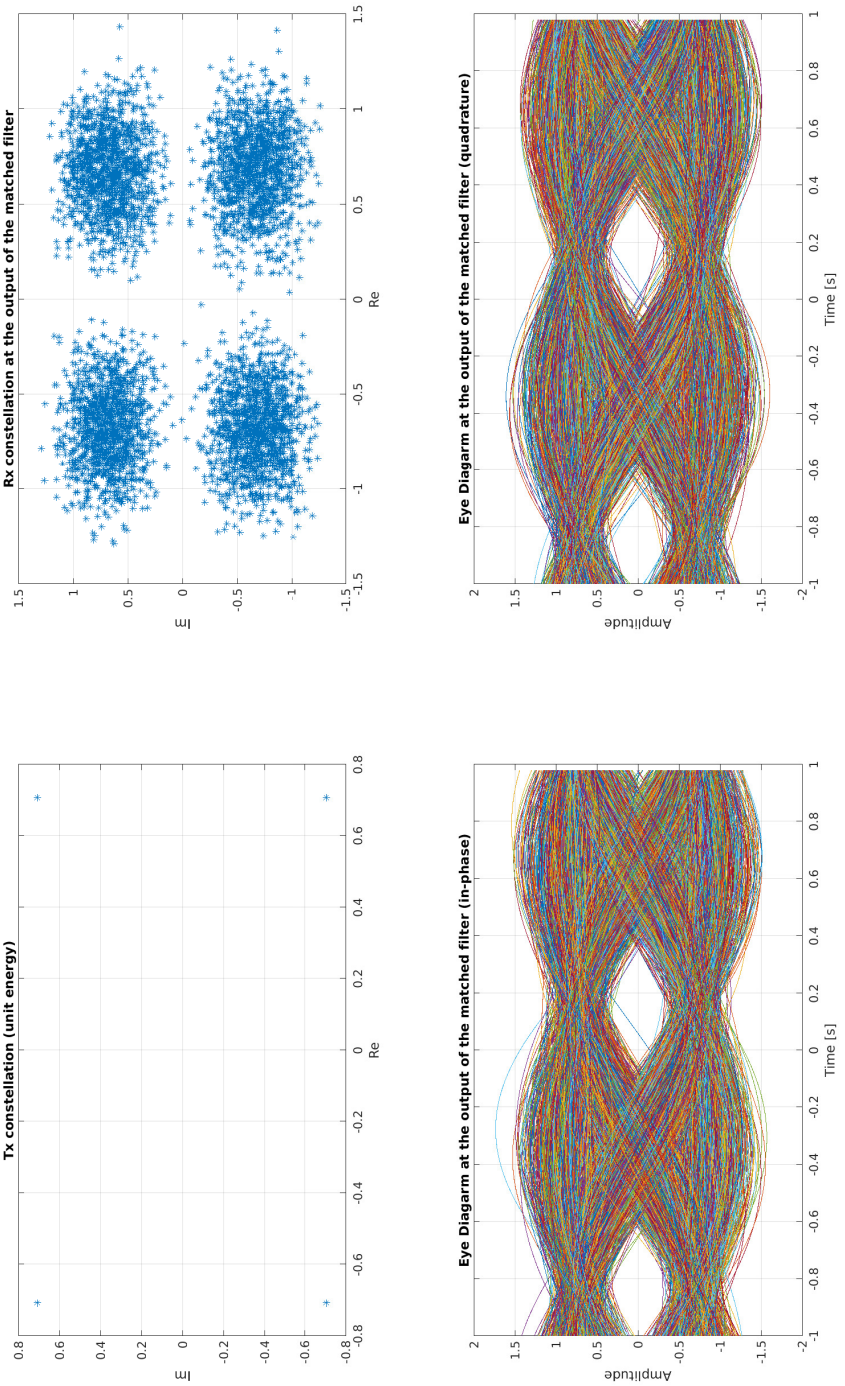


Figure 8: Constellation cloud and eye diagram for $E_s/N_0 = 15$ dB when channel introduces $d = 8$ samples of delay

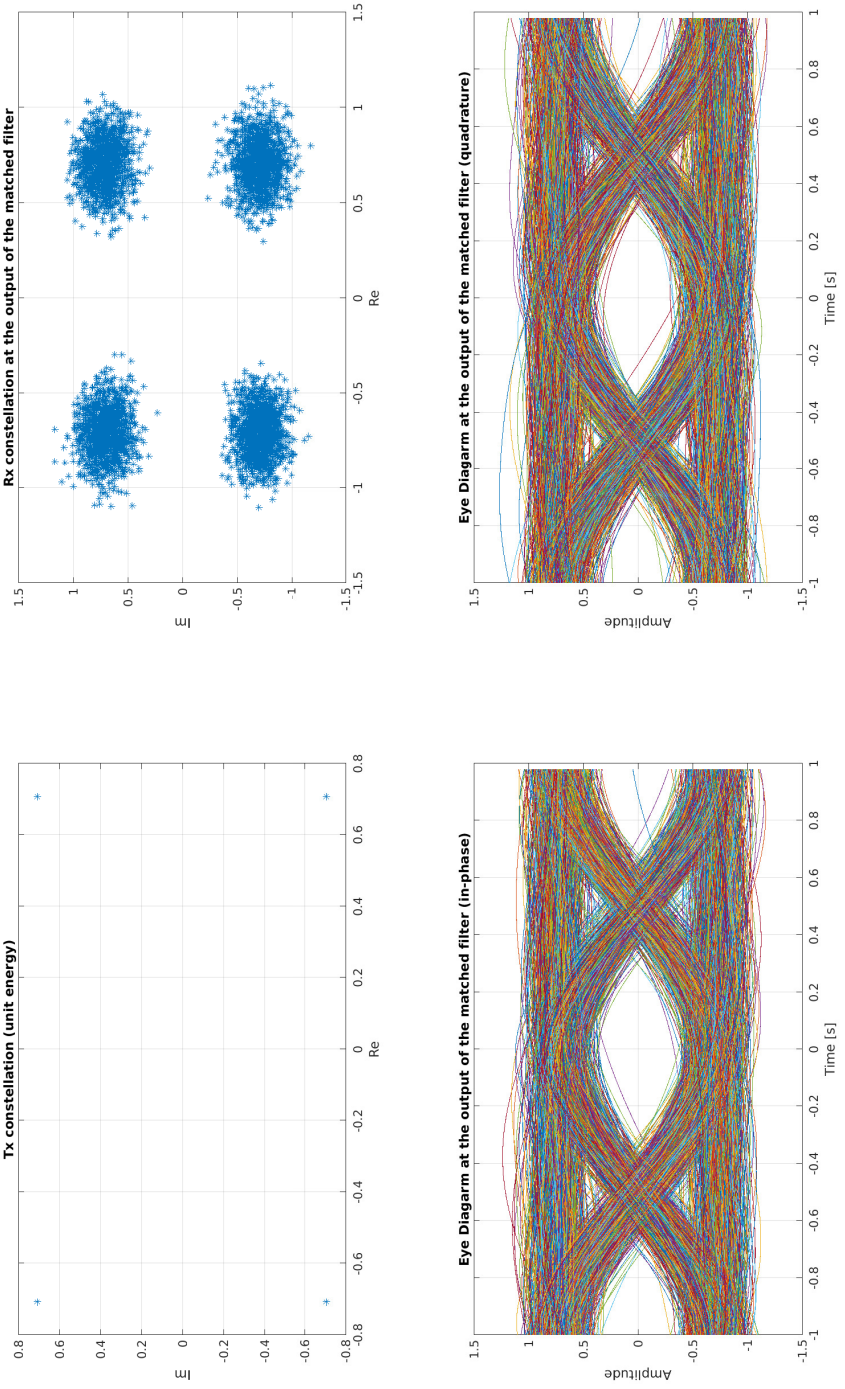


Figure 9: Constellation cloud and eye diagram for $E_s/N_0 = 15$ dB with roll-off factor $\beta = 0.9$

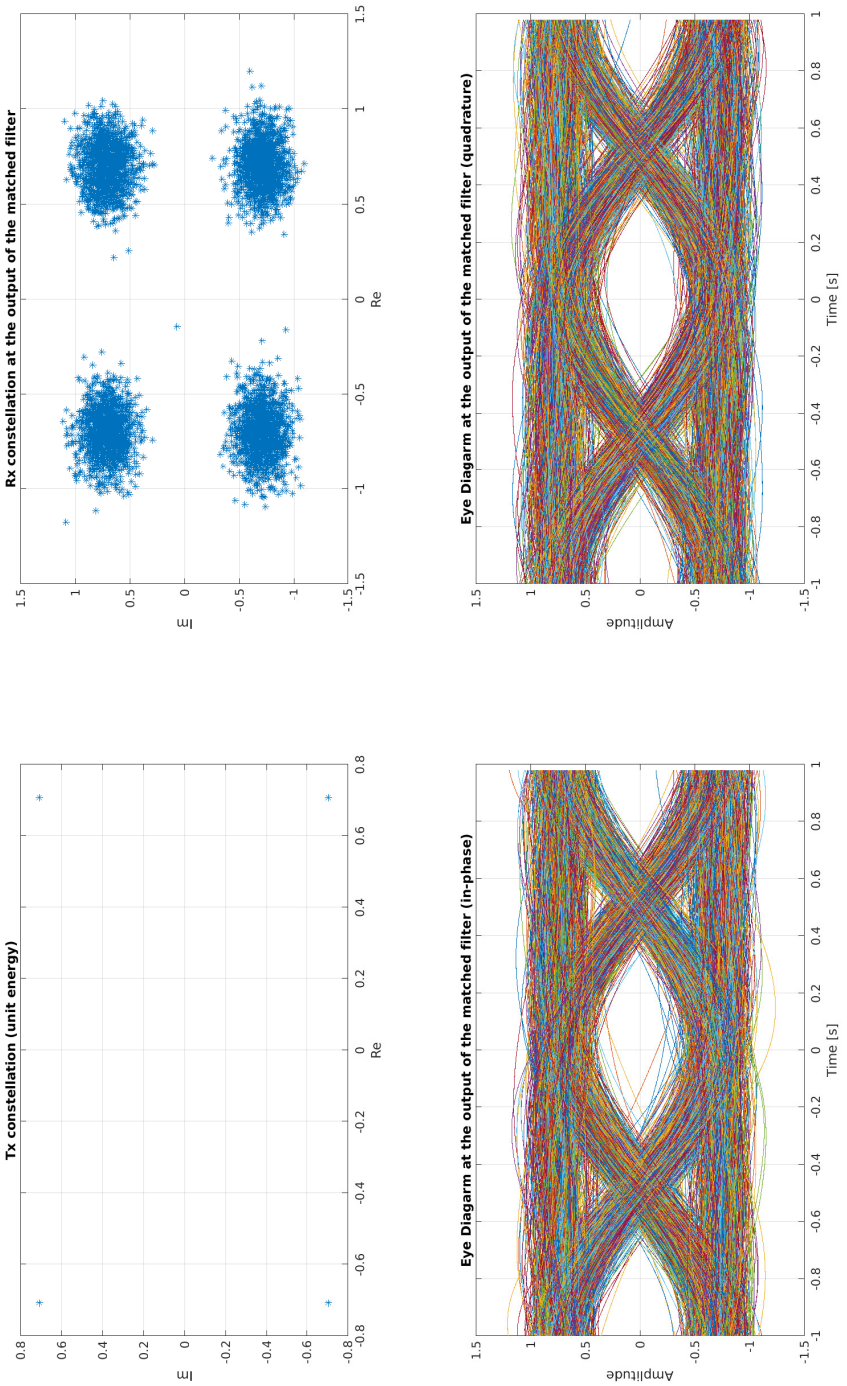


Figure 10: Constellation cloud and eye diagram for $E_s/N_0 = 15$ dB with roll-off factor $\beta = 0.9$ when the channel introduces $d = 1$ sample of delay

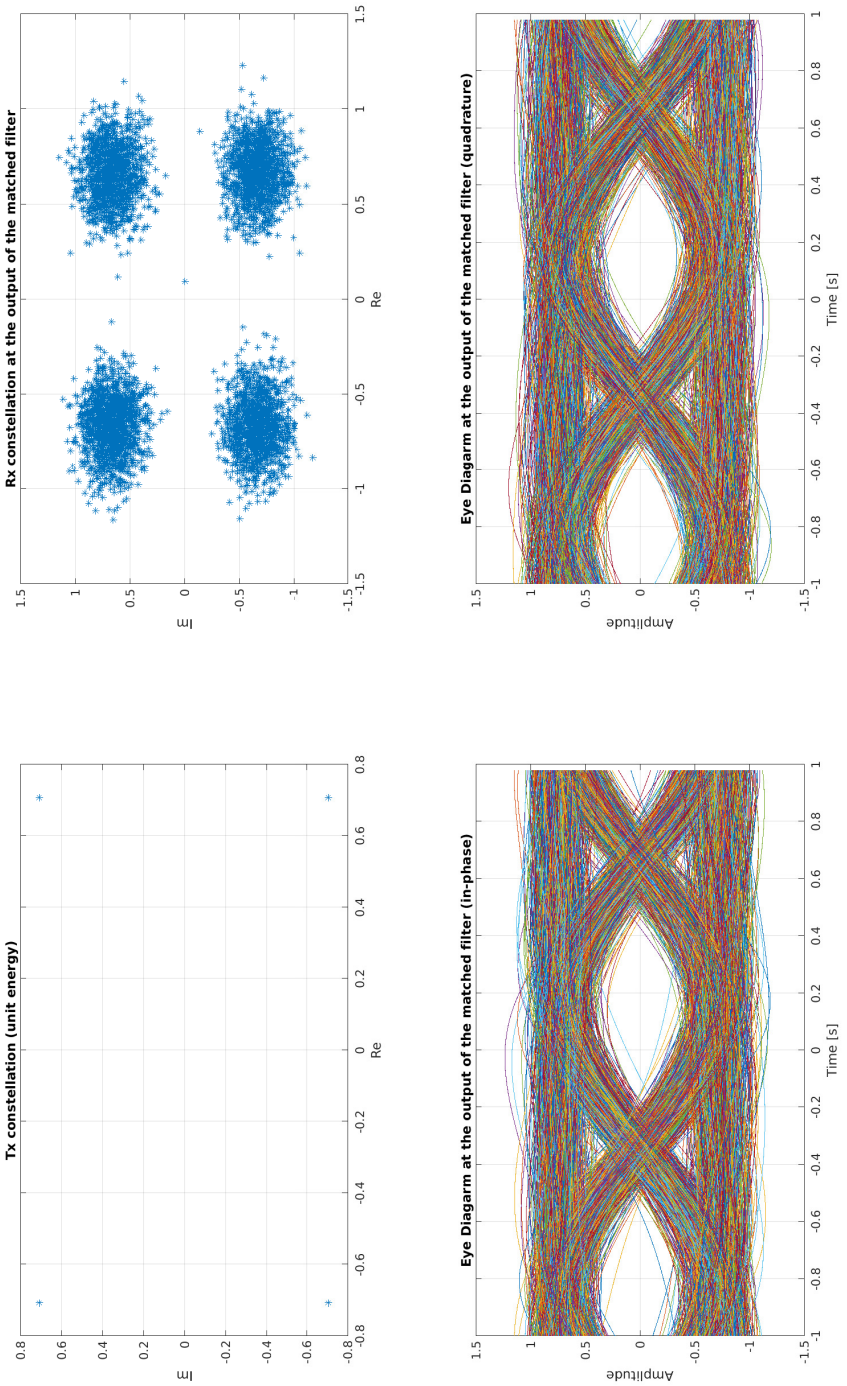


Figure 11: Constellation cloud and eye diagram for $E_s/N_0 = 15$ dB with roll-off factor $\beta = 0.9$ when the channel introduces $d = 8$ sample of delay