PROBLEM 2. 20 points (Paper and Pencil)

1. C is an $(N+L-1) \times L$ matrix:

$$C = \begin{pmatrix} x_0 & 0 & \dots & 0 \\ x_1 & x_0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ x_{N-1} & x_{N-2} & \dots & x_{N-L} \\ 0 & x_{N-1} & \dots & x_{N-L+1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & x_{N-1} \end{pmatrix}.$$

2. We are seeking for the $\hat{\boldsymbol{h}}$ which minimizes

$$\|\boldsymbol{y} - C\boldsymbol{h}\|^2$$

over all $\boldsymbol{h}$.

3. $C\hat{\boldsymbol{h}}$ is an element of the inner-product space spanned by the columns of C. So $\mathcal{V} = \text{span}\{C_1, \dots, C_L\}$ where C_i is the ith column of C, $i = 1, 2, \dots, L$.

4. We are seeking the vector $C\hat{\boldsymbol{h}} = \hat{\boldsymbol{y}} \in \mathcal{V}$ which minimizes

$$\|\boldsymbol{y} - \hat{\boldsymbol{y}}\|^2.$$

The projection theorem tells us that $\hat{\boldsymbol{y}}$ is the projection of $\boldsymbol{y}$ into $\mathcal{V}$. It has the property that the error vector $\boldsymbol{y} - \hat{\boldsymbol{y}}$ is orthogonal to every element of $\mathcal{V}$. In particular, it is orthogonal to the columns of C. Hence

$$\langle \boldsymbol{y} - C\hat{\boldsymbol{h}}, C_i \rangle = 0, \quad i = 1, 2, \dots, L.$$

Hence

$$\langle \boldsymbol{y}, C_i \rangle = \langle C\hat{\boldsymbol{h}}, C_i \rangle, \quad i = 1, 2, \dots, L.$$

Equivalently,

$$C_i^\dagger \boldsymbol{y} = C_i^\dagger C\hat{\boldsymbol{h}} \quad i = 1, 2, \dots, L.$$

In matrix form

$$C^\dagger \boldsymbol{y} = C^\dagger C\hat{\boldsymbol{h}}.$$

5. Solving for $\hat{\boldsymbol{h}}$ yields the least-squares approximation of $\boldsymbol{h}$:

$$\hat{\boldsymbol{h}} = \left(C^\dagger C\right)^{-1} C^\dagger \boldsymbol{y}.$$

6. The MMSE estimator is $\hat{\boldsymbol{h}}_{\text{MMSE}}(\boldsymbol{y}) = K_{\boldsymbol{HY}} K_{\boldsymbol{Y}}^{-1} \boldsymbol{y}$, where $K_{\boldsymbol{HY}} = E\left[\boldsymbol{HY}^\dagger\right] = K_{\boldsymbol{H}} C^\dagger$ and $K_{\boldsymbol{Y}} = E\left[\boldsymbol{YY}^\dagger\right] = CK_{\boldsymbol{H}} C^\dagger + K_{\boldsymbol{Z}}$.