ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 4 — Modern Digital Communications

Assignment 3 — September 24, 2025 (due Sept 30, 2025)

ISI and Eye Diagram

As we discussed in class, it is often desirable to communicate via a unit-norm pulse $\psi(t)$ that is orthogonal to its T-spaced shifts, i.e.,

$$\langle \psi(t), \psi(t-kT) \rangle = \mathbb{1}\{k=0\}. \tag{1}$$

Signalling via such pulses, called hereafter *Nyquist* pulses, has several advantages: As the data symbols will be the coefficients of the orthonormal expansion of the transmitted waveform, the transmitter can generate the discrete-time samples of the output signal by using a single shaping filter. At the receiver, a single matched filter does all the job required for projecting the received signal onto the basis vectors and computing the sufficient statistics for decision.

Given a Nyquist pulse $\psi(t)$, the transmitter converts the data symbols $\{s_k\}$ to the transmitted signal

$$s(t) = \sum_k s_k \psi(t-kT). \tag{2}$$

If $R(t) = s(t) + N(t)$ is the (noisy) received version of $s(t)$, the sufficient statistics for deciding the data symbol sequence is formed by projecting $R(t)$ onto the space spanned by $\{\psi(t-kT)\}_{k\in\mathbb{Z}}$. That is, to compute

$$y_k = \langle R(t), \psi(t-kT) \rangle = (R \star h_{\mathrm{MF}})(kT),$$

where $h_{\mathrm{MF}}(t) = \psi^*(-t)$ is the matched-filter impulse response.[1] Observe that by signaling via Nyquist pulses, we are able to form the decision statistics by sampling the output of a single matched filter. The samples are taken at integer multiples of T.

Nyquist's theorem translates the orthogonality condition (1) to the equivalent frequency-domain as

$$\operatorname{l.i.m.} \sum_{k=-\infty}^{\infty} \left| \psi_{\mathcal{F}}\left(f - \frac{k}{T}\right) \right|^2 = T, \qquad f \in \mathbb{R}. \tag{3}$$

Note that l. i. m. stands for *limit in mean-squared error*. Roughly speaking, it means that the summation on the left-hand-side of (3) should be equal to the constant function T, except possibly at some isolated points. In all practical applications, $\psi_{\mathcal{F}}$ is a smooth function and we ignore the l. i. m. in (3).

Nyquist criterion allows us to design a Nyquist pulse $\psi(t)$ (satisfying (1)) in the frequency domain so as to achieve a desired power spectral density (PSD).

A popular family of Nyquist pulses is the *root-raised-cosine* family, defined as:

$$|\psi_{\mathcal{F}}|^2(f) = \begin{cases} T, & |f| \le \frac{1-\beta}{2T} \\ \frac{T}{2}\left(1 + \cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right), & \frac{1-\beta}{2T} < |f| < \frac{1+\beta}{2T} \\ 0, & \text{otherwise} \end{cases}$$

$$\psi(t) = \frac{4\beta}{\pi\sqrt{T}} \frac{\cos\left((1+\beta)\pi\frac{t}{T}\right) + \frac{(1-\beta)\pi}{4\beta}\operatorname{sinc}\left((1-\beta)\frac{t}{T}\right)}{1 - \left(4\beta\frac{t}{T}\right)^2},$$

where $\beta \in (0,1)$ is called the *roll-off* factor.

Even if $\psi(t)$ is a Nyquist pulse, the sampled output of the matched filter can contain interference from other symbols, known as the *inter-symbol interference* (ISI). There are three main reasons for ISI. First, if $\psi(t)$ is a band-limited pulse then it has infinite support in time domain. Therefore, it has to be truncated to be usable in practice. A truncated version of $\psi(t)$ may not be a Nyquist pulse anymore. The second reason for ISI is *sampling-time offset*: due to a small clock offset, the receiver may sample the output of the matched filter at time $t = kT + t_0$ with $t_0 \neq 0$. Finally, the system may suffer from ISI if the channel is not *transparent* to the input signal, i.e., if its frequency response is not constant over the bandwidth of $s(t)$.

[1] We use $\star$ to denote convolution: $(x \star y)(t) := \int x(\tau) y(t-\tau)\, d\tau$.

EXERCISE 1. Consider the communication system of Figure 1. As we discussed above, the transmitter converts the sequence of data symbols $\{s_k\}$ to the transmitted signal $s(t)$ as in (2). Before adding white Gaussian noise $N(t)$ to $s(t)$, the channel filters the transmitted signal. The latter models the non-ideal behavior of the channel. The receiver, as before, filters the received signal with the matched filter $h_{\text{MF}}(t)$ and samples the matched filter output at times $t = kT$, $k \in \mathbb{Z}$ to form observables for decoding the data sequence $\{s_k\}$. $N(t)$ is assumed to be AWGN with zero mean and variance $\frac{N_0}{2}$ per real-valued dimension.

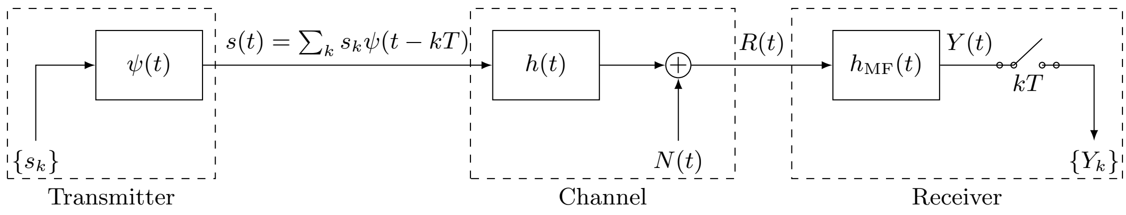
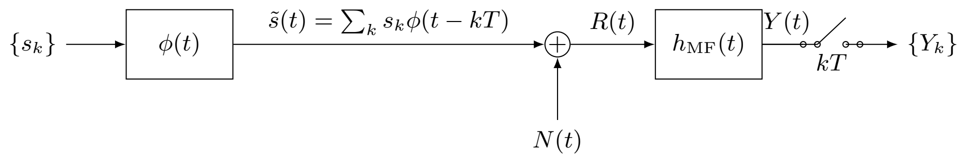


Figure 1: Communication System of Interest

1. Show that $R(t)$ has the form $R(t) = \tilde{s}(t) + N(t)$ where

$$\tilde{s}(t) = \sum_k s_k \phi(t - kT), \qquad \text{and } \phi(t) = \psi(t) \star h(t).$$

2. As a consequence, the receiver can assume the following equivalent system:



Show that the sampled matched-filter output takes the form

$$Y_k = l_0 s_k + \sum_{j \neq 0} l_j s_{k-j} + Z_k \tag{4}$$

where Z_k is additive Gaussian noise.

The term, $\sum_{j \neq 0} l_j s_{k-j}$, is known as ISI (inter-symbol interference). Derive an expression for the ISI coefficient, l_j, in terms of

$$\rho_{\phi,\psi}(t) = \int \phi(\alpha + t)\psi^*(\alpha) d\alpha.$$

Recall that $h_{\text{MF}}(t) = \psi^*(-t)$. As a sanity check, verify that if $\psi(t)$ is a Nyquist pulse with bandwidth B and the channel is an ideal low-pass filter with bandwidth greater than B, your expression for l_j yields $l_j = \mathbb{1}\{j = 0\}$.

3. Show that if $\psi(t)$ is a Nyquist pulse, the noise sequence $\{Z_k\}$ in (4) is i.i.d.

EXERCISE 2. In Exercise 1 we have seen that having a non-transparent channel response $h(t)$ results in the ISI. In this exercise, we investigate the effect of a sampling-time offset. Consider again the system of Figure 1, and now assume that the output of the matched filter is sampled at time $kT + t_0$ for some arbitrary value of t_0, i.e., $Y_k = Y(kT + t_0)$.

1. Argue that the system with sampling-time offset (i.e., $t_0 \neq 0$) and channel response $h(t)$ is equivalent to an offset-free system with channel response $h(t + t_0)$.

2. Using your results from part 2 of Exercise 1, give an expression for the ISI coefficients in the presence of sampling-time offset, when the channel has an ideal impulse response, $h(t) = \delta(t)$.

3. Recall from the class that $\psi(t) = 1/\sqrt{T}\,\text{sinc}(t/T)$ is a Nyquist pulse. Observe also that it is the minimum-bandwidth Nyquist pulse (for a given symbol period T). However, sinc$(\cdot)$ turns out to be a bad choice from the perspective of the sensitivity of the system to sampling-time offset. Let us see why. For simplicity assume $T = 1$, hence $\psi(t) = \text{sinc}(t)$. Assume also that the channel had an ideal impulse response.

   (a) Compute the ISI coefficients l_j as a function of t_0 when $\psi(t) = \text{sinc}(t)$.

   (b) Show that with this choice of $\psi(t)$ the ISI can, in principle, be unbounded.
   Hint. For arbitrary constants c and d, the sum $\sum_{j=1}^{n} \frac{c}{j+d}$ grows without bound as n increases.

Exercise 3. The *eye diagram* is a powerful tool to visualize if there is ISI in the system and how critical it is. It is obtained as follows: Suppose $Y(t)$ is the matched filter output when $s(t)$ is the transmitted signal corresponding to a sequence of random data symbols as in (2) (see Figure 1). If we plot $Y(t - kT)$ for $t \in [-T, T]$ and various integers $k \in \mathbb{Z}$ on the same axes, we obtain the *eye diagram.*

In this exercise we write a function to plot the eye diagram and use it to see the effects of ISI in a communication system.

1. Implement `my_eyediagram` (for Python: `my_utilPDC.my_eyediagram`).

   Given the discrete-time samples of the matched filter output `y`, sampled at the rate `Fs` samples per second, the function plots the traces of $Y(t - iT)$, $t \in [-T, T]$, for different values of i. Note that each trace corresponds to `2*T*Fs` samples. If the number of samples in `y` is not an integer multiple of `2*T*Fs`, the function should ignore the extra samples. The horizontal axes must be labeled in the range $[-T, T]$.

   Hint. The function `plot(t,z)` (`matplotlib.pyplot.plot(t, z)` for Python) accepts a vector `t` and a matrix `z`.

   You will use the function `my_eyediagram` in the next part of the exercise. For now, to test your function, we provide a data file, `mf_output.mat`, that contains the discrete time output of the matched filter sampled at a rate of `Fs = 5000` samples per second with the symbol period `T = 0.01` second. The signal is obtained by transmitting random BPSK data symbols via the pulse $\psi(t) = \frac{1}{\sqrt{T}}\,\text{sinc}(t/T)$ truncated to length 20. The transmission has taken place over a noiseless ideal channel.

   If you run your function with the input from the file, i.e., execute
   `>> load('mf_output.mat'); my_eyediagram(y,Fs,T)`
   or `my_plot_eye_diagram.py` for Python, you should get the eye diagram of Figure 2. By looking at Figure 2, determine whether there is ISI in the system.

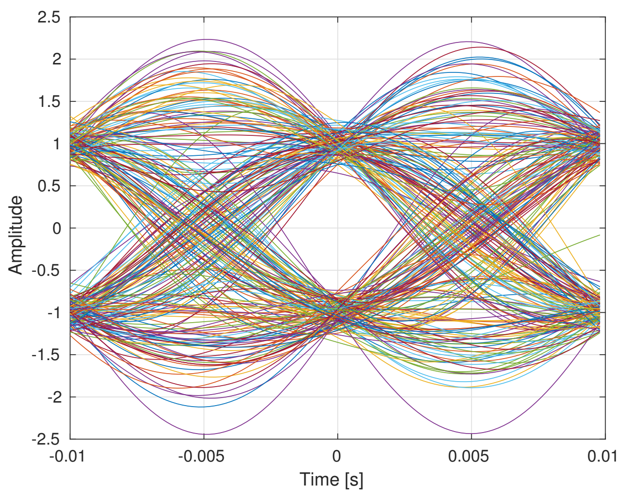


Figure 2: The Eye Diagram for sinc

In the remainder of this exercise we would like to investigate the effect of truncating the pulse. We do so using the eye diagram.

2. Modify the script `my_errorRatesScript` which you wrote in Assignment 2 such that it transmits 500 BPSK symbols. The transmission should take place over an ideal noiseless channel. The idea is to use this modified script to investigate the ISI caused from truncating root-raised-cosine pulses with different roll-off factors. Specifically, consider root-raised-cosine pulses with roll-off factors[2] $\beta = 0, 0.3, 0.9$ and examine two truncated versions of each pulse, one truncated to length $6T$, and the other one to $20T$. Assume that the data rate is $R = 100$ symbols/sec and the sampling frequency is $F_s = 5000$ Hz. For each of the RRC pulses above, plot them in the time and frequency domain, using the function `tfplot` which you wrote in Assignment 1. Then, use `my_eyediagram` to plot the eye diagram at the output of the matched filter.

   For which values of β and truncation lengths is the ISI evident?

EXERCISE 4. In this exercise we see how the eye diagram can help us in assessing the performance of a communication system.

1. Modify the script `my_errorRatesScript` which you wrote in Assignment 2 such that it uses a 4-QAM constellation and a root-raised-cosine filter with roll-off factor $\beta = 0.22$. Truncate the root-raised-cosine pulse to length $16T$. Use a symbol rate of 1 symbol per second and upsampling factor of 50. Using your script, simulate the transmission of 10^4 4-QAM symbols over the channel with $E_s/N_0 = 10$ dB. Your script should also

   - do a scatter plot of the transmitted symbols,
   - do a scatter plot of the sampled MF outputs,
   - plot the eye diagram at the matched filter output.

   Observe that, as the signal of interest is now complex, you actually need to generate two separate eye diagrams for the real (in-phase) and imaginary (quadrature) components of the signal.

   Repeat the same experiment for signal-to-noise ratios of $E_s/N_0 = 15$ dB and $E_s/N_0 = 20$ dB. Do you see a correlation between the shape of the eye and the size of the 'clouds' in the received symbols constellation?

2. Now suppose, besides adding noise to the transmitted signal, the channel *delays* the signal by d samples. (This can also model the offset in the sampling clock of the receiver, as you have shown in Exercise 2, assuming that the offset is a multiple of sampling period.) This delay can easily be modeled by inserting d zeros in front of the transmitted signal (before adding noise).

   What is the effect of the delay on the eye diagram of our system? Try a small value, like $d = 1$ and a relatively larger one, like $d = 8$ and see what happens to the eye. (For this experiment we fix the signal-to-noise ratio to $E_s/N_0 = 15$ dB.)

3. Repeat the same experiment as in part 2 but now with a root-raised-cosine pulse with roll-off factor $\beta = 0.9$. What difference do you observe?

[2] It is obvious that a root-raised-cosine pulse with roll-off factor $\beta = 0$ is $\text{sinc}(\cdot)$