Solution Set 5

Problem 1: Optimal coupling

Let X and Y be random variables on common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The total variation distance is defined as

$$d_{TV}(X,Y) = \sup_{A\in\mathcal{B}(\mathbb{R})} |(\mathbb{P}(\{X \in A\}) - \mathbb{P}(\{Y \in A\}))| .$$

a) Show that

$$\mathbb{P}(\{X = Y\}) \leq 1 - d_{TV}(X,Y).$$

b) Let X and Y be discrete and supported on $\{0,1,2,\dots\}$ with pmfs p_X and p_Y. Show that

$$d_{TV}(X,Y) = \frac{1}{2}\sum_{m\geq 0} |p_X(m) - p_Y(m)|.$$

c) (Optimal coupling) Let $\Omega = \{1,\dots k\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}(\omega) = \frac{1}{k} \quad \forall \omega \in \Omega$. Suppose that X is Bernoulli $\left(\frac{k-1}{k}\right)$ and Y is Bernoulli $\left(\frac{2}{k}\right)$. What is $d_{TV}(X,Y)$?

Construct an explicit mappings $X\colon \Omega \to \{0,1\}$ and $Y\colon \Omega \to \{0,1\}$ so that the bound in part a) is satisfied with equality.

Solution

Let X and Y be random variables on common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The total variation distance is defined as

$$d_{TV}(X,Y) = \sup_{A\in\mathcal{B}(\mathbb{R})} |(\mathbb{P}(X \in A) - \mathbb{P}(Y \in A))| .$$

a) (5 points) Show that

$$\mathbb{P}(X = Y) \leq 1 - d_{TV}(X,Y).$$

Solution: Let $A \in \mathcal{F}$ be the set that achieves the supremum.

$$\begin{aligned}\mathbb{P}(\{X \neq Y\}) &\geq \mathbb{P}(\{X \in A, Y \notin A\}) = \mathbb{P}(\{X \in A\}) - \mathbb{P}(\{X \in A, Y \in A\}) \\ &\geq \mathbb{P}(\{X \in A\}) - \mathbb{P}(\{Y \in A\}) = d_{TV}(X,Y)\end{aligned}$$

This shows the desired result.

(Note, if no A that achieves the above supremum exists, we can take a set A that is arbitrarily close to achieving the supremum, and the same argument will hold with minor modification.)

b) (5 points) Let X and Y be discrete and supported on $\{0,1,2,\dots\}$ with pmfs p_X and p_Y. Show that

$$d_{TV}(X,Y) = \frac{1}{2}\sum_{m\geq 0} |p_X(m) - p_Y(m)|.$$

Solution: First, note that if $A \in \mathcal{F}$ achieves the supremum above, so does $\bar{A} \in \mathcal{F}$. Let $A = \{m \colon \mathbb{P}(\{X = m\}) > \mathbb{P}(\{Y = m\})\}$. We show that A achieves the supremum in the definition above. Take any other set $B \in \mathcal{F}$ and assume without loss of generality that $\mathbb{P}(\{X \in B\}) \geq \mathbb{P}(\{Y \in B\})$. Then

$$\begin{aligned}
&\mathbb{P}(\{X \in B\}) - \mathbb{P}(\{Y \in B\}) \\
&= \mathbb{P}(\{X \in A \cap B\}) + \mathbb{P}(\{X \in \bar{A} \cap B\}) - (\mathbb{P}(\{Y \in A \cap B\}) + \mathbb{P}(\{Y \in \bar{A} \cap B\})) \\
&= \mathbb{P}(\{X \in A \cap B\}) - \mathbb{P}(\{Y \in A \cap B\}) + \mathbb{P}(\{X \in \bar{A} \cap B\}) - \mathbb{P}(\{Y \in \bar{A} \cap B\})) \\
&\leq \mathbb{P}(\{X \in A \cap B\}) - \mathbb{P}(\{Y \in A \cap B\}) \\
&\leq \mathbb{P}(\{X \in A\}) - \mathbb{P}(\{Y \in A\})
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{1}{2} \sum_{m \geq 0} |p_X(m) - p_Y(m)| &= \frac{1}{2} \sum_{m \in A} (p_X(m) - p_Y(m)) + \frac{1}{2} \sum_{m \in \bar{A}} (p_Y(m) - p_X(m)) \\
&= \frac{1}{2} d_{TV}(X,Y) + \frac{1}{2} d_{TV}(X,Y) = d_{TV}(X,Y)
\end{aligned}$$

c) (4 points) (Optimal coupling) Let $\Omega = \{1, \dots k\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}(\omega) = \frac{1}{k} \quad \forall \omega \in \Omega$. Suppose that X is Bernoulli $\left(\frac{k-1}{k}\right)$ and Y is Bernoulli $\left(\frac{2}{k}\right)$. What is $d_{TV}(X,Y)$?

Construct an explicit mappings $X \colon \Omega \to \{0,1\}$ and $Y \colon \Omega \to \{0,1\}$ so that the bound in part a) is satisfied with equality.

Solution: By using the formula in part b) we see that

$$d_{TV}(X,Y) = \frac{1}{2} \left(\left| \frac{k-1}{k} - \frac{2}{k} \right| + \left| \frac{1}{k} - \frac{k-2}{k} \right| \right) = \frac{k-3}{k}$$

One possible mapping that achieves this is

$$X(\omega) = \begin{cases} 1 & \omega \in \{1, 2, \dots, k-1\} \\ -1 & \text{otherwise} \end{cases}$$

and

$$Y(\omega) = \begin{cases} 1 & \omega \in \{1, 2\} \\ -1 & \text{otherwise} \end{cases}$$

In this case $X(\omega) = Y(\omega)$ for $\omega \in \{1, 2, k\}$.

Problem 2: Poisson coupling

a) Show that a sum of two independent Poisson random variables is a Poisson random variable.

b) Let X be a Poisson random variable with parameter λ and Y a Poisson random variable with parameter ν, $\lambda > \nu$. Use the coupling inequality to show that

$$d_{TV}(X,Y) \leq \lambda - \nu.$$

Hint: Use part a) to construct a coupling of X and Y.

Solution a) Let X be a Poisson random variable with parameter λ and Y a Poisson random variable with parameter ν, for some $\lambda, \nu \geq 0$. Then,

$$\begin{aligned} P(X+Y=k) &= \sum_{i=0}^{k} P(X+Y=k, X=i) \\ &= \sum_{i=0}^{k} P(Y=k-i, X=i) \\ &= \sum_{i=0}^{k} P(Y=k-i)P(X=i) \\ &= \sum_{i=0}^{k} e^{-\nu}\frac{\nu^{k-i}}{(k-i)!}e^{-\lambda}\frac{\lambda^i}{i!} \\ &= e^{-(\nu+\lambda)}\frac{1}{k!}\sum_{i=0}^{k}\frac{k!}{i!(k-i)!}\nu^{k-i}\lambda^i \\ &= e^{-(\nu+\lambda)}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\nu^{k-i}\lambda^i \\ &= \frac{(\nu+\lambda)^k}{k!}\cdot e^{-(\nu+\lambda)} \end{aligned}$$

Thus, $X+Y$ is a Poisson random variable with parameter $\lambda+\nu$.

b) Construct the coupling $\hat{X} = \hat{Y} + \hat{Z}$ with $\hat{Z}$ being Poisson distributed with parameter $\lambda - \nu$. Then,

$$\begin{aligned} d_{TV}(X,Y) &= \inf\{(\hat{X} \neq \hat{Y}) : \text{couplings}(\hat{X}, \hat{Y}) \text{ of } (X,Y)\} \\ &\leq \mathbb{P}(\hat{X} \neq \hat{Y}) \\ &= \mathbb{P}(\hat{Y} + \hat{Z} \neq \hat{Y}) \\ &= \mathbb{P}(\hat{Z} \neq 0) \\ &= 1 - \mathbb{P}(\hat{Z} \neq 0) \\ &= 1 - e^{-(\lambda-\nu)}. \end{aligned}$$

From Taylor expansion, we know that $e^{-(\lambda-\nu)} \geq 1 - (\lambda - \nu)$. This leads to

$$d_{TV}(X,Y) \leq 1 - e^{-(\lambda-\nu)} \leq \lambda - \nu.$$