Advanced Probability & Applications
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Solution Set 4

Problem 1: Expecation and CDF

a) Let X be a *continuous* and *non-negative* random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Show that

$$\mathbb{E}(X) = \int_0^{+\infty} (1 - F_X(t))\, dt.$$

Extend this formula to further to any continuous random variable X. That is, show that

$$\mathbb{E}(X) = \int_0^{+\infty} (1 - F_X(t))\, dt - \int_{-\infty}^{0} F_X(t)\, dt.$$

b) Use this formula to compute $\mathbb{E}(X)$ for $X \sim \text{Laplace}(0, \lambda^{-1})$ for $\lambda > 0$.

c) Let X be a *discrete* and *non-negative* random variable taking values in $\mathbb{N}$ only. Show that

$$\mathbb{E}(X) = \sum_{k \geq 0} (1 - F_X(k)).$$

and use this new formula to compute $\mathbb{E}(X)$ when $X \sim \text{Geom}(p)$ for some $0 < p < 1$.

Solution a) Since X is continuous, it has a pdf p_X. We can write

$$1 - F_X(t) = \mathbb{P}(\{X > t\}) = \int_t^{+\infty} p_X(s)\, ds$$

and

$$\int_0^{\infty} (1 - F_X(t))\, dt = \int_0^{\infty} \left(\int_t^{+\infty} p_X(s)\, ds \right) dt = \int_0^{\infty} \left(\int_0^{s} p_X(s)\, dt \right) ds$$

where the second equality follows by exchanging integration order. Then

$$\int_0^{\infty} (1 - F_X(t))\, dt = \int_0^{\infty} [t p_X(s)]_0^s\, ds = \int_0^{\infty} s p_X(s) ds = \mathbb{E}(X)$$

For any continuous random variable we can write

$$\mathbb{E}(X) = \int_0^{\infty} s p_X(s) ds + \int_{-\infty}^{0} s p_X(s) ds$$

and it remains to show that

$$\int_{-\infty}^{0} s p_X(s) ds = - \int_{-\infty}^{0} F_X(s)\, ds$$

Following a similar sequence of steps

$$\int_{-\infty}^{0} F_X(s)\, ds = \int_{-\infty}^{0} \left(\int_{-\infty}^{s} p_X(t)\, dt \right) ds = \int_{-\infty}^{0} \left(\int_t^{0} p_X(t)\, ds \right) dt$$
$$= \int_{-\infty}^{0} [s p_X(t)]_t^0\, dt = - \int_{-\infty}^{0} t p_X(t) dt$$

b)

$$\begin{aligned}\mathbb{E}(X) &= \int_0^{\infty} \frac{1}{2}\exp(-\lambda t)dt - \int_{-\infty}^{0} \frac{1}{2}\exp(\lambda t)dt \\ &= \left[-\frac{1}{\lambda}\frac{1}{2}\exp(-\lambda t)\right]_0^{\infty} - \left[\frac{1}{\lambda}\frac{1}{2}\exp(\lambda t)\right]_{-\infty}^{0} = \frac{1}{2\lambda} - \frac{1}{2\lambda} = 0\end{aligned}$$

c)

$$\begin{aligned}\mathbb{E}(X) &= \sum_{k\geq 0} k\mathbb{P}(\{X=k\}) = \sum_{k\geq 0} k\left(\mathbb{P}(\{X>k-1\}) - \mathbb{P}(\{X>k\})\right) \\ &= \sum_{k\geq 1}\left(k\mathbb{P}(\{X>k-1\}) - (k-1)\mathbb{P}(\{X>k-1\})\right) \\ &= \sum_{k\geq 1}\mathbb{P}(\{X>k-1\}) = \sum_{k\geq 0}\left(1-F_X(k)\right)\end{aligned}$$

For $X \sim \text{Geom}(p)$ we have

$$\mathbb{E}(X) = \sum_{k\geq 0}(1-F_X(k)) = \sum_{k\geq 0}(1-p)^{k+1} = \sum_{k\geq 0}(1-p)(1-p)^k = \frac{1-p}{p}$$

Problem 2: Expectation and exponential random variable

Let $\lambda > 0$ and $X \sim \mathcal{E}(\lambda)$, and let us define $Y = X^a$, where $a \in \mathbb{R}$.

a) For what values of $a \in \mathbb{R}$ does it hold that $\mathbb{E}(Y) < +\infty$?

b) For what values of $a \in \mathbb{R}$ does it hold that $\mathbb{E}(Y^2) < +\infty$?

c) For what values of $a \in \mathbb{R}$ is $\text{Var}(Y)$:

c1) well-defined and finite? c2) well-defined but infinite? c3) ill-defined?

d) Compute $\mathbb{E}(Y)$ and $\text{Var}(Y)$ for the values of $a \in \mathbb{Z}$ such that these quantities are well-defined.

Hint: Use integration by parts, recursively.

Solution

a) We have

$$\mathbb{E}(Y) = \mathbb{E}(X^a) = \int_0^{+\infty} x^a\,\lambda\,\exp(-\lambda x)\,dx < +\infty \quad \text{if and only if} \quad a > -1$$

b) Likewise:

$$\mathbb{E}(Y^2) = \mathbb{E}(X^{2a}) = \int_0^{+\infty} x^{2a}\,\lambda\,\exp(-\lambda x)\,dx < +\infty \quad \text{if and only if} \quad a > -\frac{1}{2}$$

c) Therefore, c1) $\text{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$ is well defined and finite $\forall a > -\frac{1}{2}$; c2) $\text{Var}(Y)$ is well defined but takes the value $+\infty$ for $-\frac{1}{2} \geq a > -1$, and c3) $\text{Var}(Y)$ is ill-defined (indetermination of the type $\infty - \infty$) for $a \leq -1$.

d) The only integer values of a for which $\mathbb{E}(Y)$ and $\text{Var}(Y)$ are well-defined are non-negative values. For $a = 0$, we have $Y = X^0 = 1$, so $\mathbb{E}(Y) = 1$ and $\text{Var}(Y) = 0$. For $a \geq 1$, we obtain by integration by parts:

$$\mathbb{E}(Y) = \mathbb{E}(X^a) = \int_0^{+\infty} x^a \, \lambda \, \exp(-\lambda x) \, dx$$
$$= \int_0^{+\infty} \frac{a}{\lambda} \, x^{a-1} \, \lambda \, \exp(-\lambda x) \, dx = \ldots = \frac{a!}{\lambda^a} \cdot 1$$

so

$$\mathbb{E}(Y^2) = \mathbb{E}(X^{2a}) = \frac{(2a)!}{\lambda^{2a}} \quad \text{and} \quad \mathrm{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = \frac{(2a)! - (a!)^2}{\lambda^{2a}}$$

Problem 3: Covariance

Let X be a random variable that is symmetrically distributed (i.e. $X \sim -X$) and square-integrable with $\mathrm{Var}(X) = 1$. Let also $Y = 1_{\{X \geq 0\}}$.

a) Show that for any distribution of the random variable X, $\mathrm{Cov}(X, Y) \geq 0$.

b) Using the inequality $\mathrm{Cov}(X, Y) \leq \sqrt{\mathrm{Var}(X)} \sqrt{\mathrm{Var}(Y)}$ (whose proof is to come in the sequel of the course), find the least value $C > 0$ such that $\mathrm{Cov}(X, Y) \leq C$ for every distribution of X.

c) Compute $\mathrm{Cov}(X, Y)$ for $X \sim \mathcal{N}(0, 1)$.

d) Is it possible to find a distribution for X such that $\mathrm{Cov}(X, Y) = C$? If not, is it possible to find a sequence of random variables $(X_n, \, n \geq 1)$ with varying distributions (all respecting the above constraints) and $Y_n = 1_{\{X_n \geq 0\}}$, such that $\mathrm{Cov}(X_n, Y_n) \underset{n \to \infty}{\to} C$?

e) Is it possible to find a distribution for X such that $\mathrm{Cov}(X, Y) = 0$? If not, is it possible to find a sequence of random variables $(X_n, \, n \geq 1)$ with varying distributions (all respecting the above constraints) and $Y_n = 1_{\{X_n \geq 0\}}$, such that $\mathrm{Cov}(X_n, Y_n) \underset{n \to \infty}{\to} 0$?

Solution First note that as $X \sim -X$, it holds that $\mathbb{P}(\{X \geq 0\}) \geq \frac{1}{2}$ and $\mathbb{E}(X) = 0$.

a) $\mathrm{Cov}(X, Y) = \mathbb{E}(X \, 1_{\{X \geq 0\}}) \geq 0$ as $X \, 1_{\{X \geq 0\}}$ is a non-negative random variable.

b) Using the suggested inequality, we find

$$\mathrm{Cov}(X, Y) \leq \sqrt{\mathrm{Var}(X)} \sqrt{\mathrm{Var}(Y)} = \sqrt{1} \sqrt{\mathbb{P}(\{X \geq 0\}) - \mathbb{P}(\{(X \geq 0\})^2} \leq \sqrt{\frac{1}{4}} = \frac{1}{2} = C$$

as $\mathbb{P}(\{X \geq 0\}) - \mathbb{P}(\{(X \geq 0\})^2 \leq \frac{1}{4}$ (which is maximized when $\mathbb{P}(\{X \geq 0\}) = \frac{1}{2}$).

c) The computation gives

$$\mathrm{Cov}(X, Y) = \mathbb{E}(X \, 1_{\{X \geq 0\}}) = \int_0^{+\infty} x \, \frac{1}{\sqrt{2\pi}} \, \exp(-x^2/2) \, dx = \frac{1}{\sqrt{2\pi}} \left(-\exp(-x^2/2)\right)\Big|_{x=0}^{x=+\infty} = \frac{1}{\sqrt{2\pi}}$$

(clearly satisfying the above two inequalities)

d) The answer to the first question is yes: take X such that $\mathbb{P}(\{X = +1\}) = \mathbb{P}(\{X = -1\}) = \frac{1}{2}$ (verifying $X \sim -X$, $\mathrm{Var}(X) = 1$ and $\mathrm{Cov}(X, Y) = \frac{1}{2}$).

e) The answer to the first question is no, but the one to the second is yes: consider X_n such that $\mathbb{P}(\{X_n = n\}) = \mathbb{P}(\{X_n = -n\}) = \frac{1}{2n^2}$ and $\mathbb{P}(\{X_n = 0\}) = 1 - \frac{1}{n^2}$. Then $X_n \sim -X_n$ and $\mathrm{Var}(X_n) = 1$ for every n, and $\mathrm{Cov}(X_n, Y_n) = \mathbb{E}(X_n \, 1_{\{X_n \geq 0\}}) = n \, \frac{1}{2n^2} = \frac{1}{2n} \underset{n \to \infty}{\to} 0$.