Solution Set 1

Problem 1: Size of Infinity

Let $\mathbb{Q}$ denote rational numbers and $\mathbb{R}$ denote real numbers.

a) Show that $\mathbb{Q}$ is countable.

b) Show that every infinite subset of a countable set A is countable.

Remark: Roughly speaking, countable sets represent the "smallest" infinity.

c) Let the set A be all sequences whose elements are the digits 0 and 1. Show that A is not countable. Conclude that $\mathbb{R}$ is not countable.

d) Are irrational numbers, e.g. $\mathbb{R} \setminus \mathbb{Q}$, countable? Why or why not?

e) Construct a set that is infinite, but does not have the same cardinal number as $\mathbb{Q}$ or $\mathbb{R}$.

Solution

a) We can be more general and show that a countable union of countable sets is countable. That is, let $E_n, n = 1, 2, 3, \dots$, be a sequence of countable sets and put

$$A = \bigcup_{n=1}^{\infty} E_n.$$

Every set E_n can be arranged in a sequence $\{x_{n,k}\}, k = 1, 2, 3, \dots$. Moreover, we can construct an infinite array

$$\begin{array}{cccc} x_{11} & x_{12} & x_{13} & \cdots \\ x_{21} & x_{22} & x_{23} & \cdots \\ x_{31} & x_{32} & x_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

The array contains elements of A which can be numbered as $x_{11}; x_{21}, x_{31}, x_{22}, x_{13}; \dots$, and so on. We can likewise show that a union of two countable sets is countable.

Observe that $\mathbb{Q}$ is a special case of a countable union of countable sets since for every positive $r \in \mathbb{Q}$ we can write $r = \frac{m}{n}$ for some positive integers m and n. Thus, we can construct a similar array with the m indexing the rows, and n indexing the columns (and skipping over any duplicates). By this argument, the positive rational numbers are at most countable. They are not finite since they include positive integers as a subset. Likewise, non-positive rational numbers are also countable. Finally, $\mathbb{Q}$ is countable since it is just a union of two countable sets.

b) Let $E \in A$ and suppose that E is infinite. Since A is countable, we can arrange its elements in a sequence $\{x_n\}$ of distinct elements. Construct a new sequence $\{n_k\}$ by letting n_1 be the smallest integer such that $x_{n_1} \in E$, n_2 the next smallest, and so on. Putting $f(k) = x_{n_k}, k = 1, 2, \dots$, we obtain a 1-1 correspondence between E an positive integers.

c) The elements of A are sequences like $1, 0, 0, 1, 0, 1, 1, 1, \ldots$. Suppose A is countable, and let $s_1, s_2, s_3, \ldots$ be the sequence of all elements of A. We construct a new sequence s as follows. If the nth digit of s_n is 1, we let the nth digits of s be 0, and vice versa. Thus s is not equal to any element in the sequence, and $s \notin A$. This is a contradiction and so A is not countable.

If we represent real numbers on the interval $[0, 1]$ with their binary expansion, we get exactly the set A. Thus, the interval $[0, 1]$ is not countable. The interval $[0, 1]$ is an infinite subset of $\mathbb{R}$ and therefore (by part b), $\mathbb{R}$ is not countable.

d) Assume that the irrational numbers are countable. Then, $\mathbb{R}$ can be represented as a union of two countable sets and is countable (by part a). We have already shown that $\mathbb{R}$ is not countable in part c) so this is a contradiction. Therefore, irrational numbers are not countable.

e) One way to solve this problem is to find a set that is larger than $\mathbb{Q}$ but smaller than $\mathbb{R}$. This is a famous problem known as *the continuum hypothesis* and it is outside of the scope of this class! However, it is easy to construct an infinite set that is strictly larger that $\mathbb{R}$. Let A be the power set of $\mathbb{R}$, that is, A is the set of all subsets of R. The fact that A and $\mathbb{R}$ have different cardinal numbers can be shown by the same diagonal process argument that we used in part c) of exercise 1.1.

Problem 2: Atoms

Let $\Omega = \{1, \ldots, 6\}$ and $\mathcal{A} = \{\{1, 2, 3\}, \{1, 3, 5\}\}$.

a) Describe $\mathcal{F} = \sigma(\mathcal{A})$, the σ-field generated by $\mathcal{A}$.

Hint: For a finite set Ω, the number of elements of a σ-field on Ω is always a power of 2.

b) Give the list of non-empty elements G of $\mathcal{F}$ such that

$$\text{if } F \in \mathcal{F} \text{ and } F \subset G, \text{ then } F = \emptyset \text{ or } G.$$

These elements are called the *atoms* of the σ-field $\mathcal{F}$ (cf. course). Equivalently, an event $G \in \mathcal{F}$ is *not* an atom if there exists $F \in \mathcal{F}$ such that $F \neq \emptyset$, $F \subset G$ and $F \neq G$.

The atoms of a $\mathcal{F}$ form a *partition* of the set Ω and they also generate the σ-field $\mathcal{F}$ in this case. (note also that if m is the number of atoms of $\mathcal{F}$, then the number of elements of $\mathcal{F}$ equals 2^m)

c) Let $X_1(\omega) = 1_{\{1,2,3\}}(\omega)$, $X_2 = 1_{\{1,3,5\}}(\omega)$ and $Y(\omega) = X_1(\omega) + X_2(\omega)$. Does it hold that $\sigma(Y) = \sigma(X_1, X_2)$?

Solution

a) $\mathcal{F} = \{\emptyset, \{2\}, \{5\}, \{1, 3\}, \{2, 5\}, \{4, 6\}, \{1, 2, 3\}, \{1, 3, 5\}, \{2, 4, 6\}, \{4, 5, 6\},$
$\{1, 2, 3, 5\}, \{1, 3, 4, 6\}, \{2, 4, 5, 6\}, \{1, 2, 3, 4, 6\}, \{1, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}$ ($16 = 2^4$ elements)

b) atoms of $\mathcal{F}$: $\{1, 3\}, \{2\}, \{4, 6\}, \{5\}$. Notice that one also has $\mathcal{F} = \sigma(\{1, 3\}, \{2\}, \{4, 6\}, \{5\})$, as already mentioned in the problem set.

c) Nearly by definition, $\sigma(X_1, X_2) = \sigma(\{1, 2, 3\}, \{1, 3, 5\}) = \mathcal{F}$. Besides, the random variable Y satisfies: $Y(1) = Y(3) = 2$, $Y(2) = Y(5) = 1$ and $Y(4) = Y(6) = 0$. We deduce from there that the atoms of $\sigma(Y)$ are $\{1, 3\}$, $\{2, 5\}$ and $\{4, 6\}$, and therefore that Y contains less information than X_1, X_2, i.e., that $\sigma(Y) \subset \sigma(X_1, X_2)$ and $\sigma(Y) \neq \sigma(X_1, X_2)$.

Problem 3: Borel atoms

Let now $\Omega = [0,1]$ and $\mathcal{F} = \mathcal{B}([0,1])$ be the Borel σ-field on $[0,1]$.

a) What are the atoms of $\mathcal{F}$?

b) Is it true in this case that the σ-field $\mathcal{F}$ is generated by its atoms?

c) Describe the σ-field $\sigma(\{x\}, x \in [0,1])$.

Solution

a) The atoms of $\mathcal{F}$ are the singletons $\{x\}$, with $x \in [0,1]$.

b) The answer is no. One can check indeed that the σ-field generated by the sets $\{x\}, x \in [0,1]$ is the list of all countable subsets of $[0,1]$, as well as all the complements of countable subsets of $[0,1]$, which is of course not equal to the list of all Borel subsets of $[0,1]$. In particular, the open intervals are not in the list.

c) $\sigma(\{x\}, x \in [0,1])$ comprises all countable unions of singletons in $[0,1]$, as well as all the complements of these sets. One can check that indeed, such a collection of sets is a σ-field, which is moreover *much* smaller than $\mathcal{B}([0,1])$.

Problem 4: Sigma-filed generated by atoms

Let Ω be an arbitrary set and $\mathcal{F}$ be a $\sigma-$ field on Ω. In this problem we will show that if $\mathcal{F}$ is infinite, it must be uncountable. We will proceed with proof by contradiction and assume that $\mathcal{F}$ is countable.

a) For every $\omega \in \Omega$, define $B_\omega = \bigcap_{A \in \mathcal{F}:\, \omega \in A} A$. Is $B_\omega \in \mathcal{F}$? Why or why not?

b) Let $\mathcal{C} = \{B_\omega\}_{\omega \in \Omega}$ be a collection of all such unique B_ω. Argue that $\mathcal{C}$ partitions Ω and that it is at most finite, or countable.

c) Argue that $\sigma(\mathcal{C}) = \mathcal{F}$. That is, the σ-field generated by $\mathcal{C}$ is exactly $\mathcal{F}$.

d) Conclude from parts (a) - (c) that there is a contradiction and it is not possible for $\mathcal{F}$ to be countable.

Solution

Let Ω be an arbitrary set and $\mathcal{F}$ be a $\sigma-$ field on Ω. In this problem we will show that if $\mathcal{F}$ is infinite, it must be uncountable. We will proceed with proof by contradiction and assume that $\mathcal{F}$ is countable.

a) For every $\omega \in \Omega$, define $B_\omega = \bigcap_{A \in \mathcal{F}:\, \omega \in A} A$. Is $B_\omega \in \mathcal{F}$? Why or why not?

Answer: We have assumed that $\mathcal{F}$ is countable. Thus, the collection of all the sets containing ω i.e., $S_\omega = \{A : \omega \in A\}$ can be at most countable, as $S_\omega \subset \mathcal{F}$. Further, note that the countable intersection of sets in $\mathcal{F}$ is also an element of $\mathcal{F}$. Thus, $B_\omega := \cap S_\omega$ is an element of $\mathcal{F}$.

b) Let $\mathcal{C} = \{B_\omega\}_{\omega \in \Omega}$ be a collection of all such unique B_ω. Argue that $\mathcal{C}$ partitions Ω and that it is at most finite, or countable.

Answer: To show that B_ω partitions $\mathcal{F}$ we need to show that: 1)$\forall \omega_1, \omega_2 \in \Omega$, we have $B_{\omega_1} \cap B_{\omega_2} = \emptyset$ or $B_{\omega_1} = B_{\omega_2}$, 2) that $\cup_{\omega \in \Omega} B_\omega = \Omega$.

1) Suppose there exists $\omega_2 \in B_{\omega_1}$ such that $B_{\omega_1} \neq B_{\omega_2}$. Then, $B_{\omega_1} \cap B_{\omega_2}$ is a strict subset of B_{ω_2} or

it is exactly B_{ω_2}. In the first case, it contradicts the fact that B_{ω_2} is the smallest set in $\mathcal{F}$ containing ω_2. In the second case, it means that B_{ω_2} is a proper subset of B_{ω_1} which again contradicts the fact that B_{ω_1} is the smallest set in $\mathcal{F}$ containing ω_1. Indeed, either $\omega_1 \in B_{\omega_2}$ or $\omega_1 \in B_{\omega_1} \cap B^c_{\omega_2}$.

2) Since every $\omega \in \Omega$ is in some B_ω, $\cup_{\omega\in\Omega} B_\omega = \Omega$.

Since $\mathcal{F}$ is countable, and $\mathcal{C}$ is a subset of $\mathcal{F}$ it is either countable or finite.

c) Argue that $\sigma(\mathcal{C}) = \mathcal{F}$. That is, the σ-field generated by $\mathcal{C}$ is exactly $\mathcal{F}$.

Answer:

For any $A \in \mathcal{F}$ we can show that $A = \cup_{\omega\in A} B_\omega$. Indeed, $A \subset \cup_{\omega\in A} B_\omega$ is trivial. We can show that $\cup_{\omega\in A} B_\omega \subset A$ by a similar argument as in part b). Assume that there exists $\omega_1 \in \cup_{\omega\in A} B_\omega$ such that $\omega_1 \notin A$. But then, either $B_{\omega_1} \cap A = \emptyset$ or $B_{\omega_1} \cap A$ is a proper subset of B_{ω_1} which again contradicts the minimality of B_{ω_1} for some $\omega_2 \in B_{\omega_1} \cap A$.

d) Conclude from parts (a) - (c) that there is a contradiction and it is not possible for $\mathcal{F}$ to be countable.

Answer: Observe that we have shown that $\mathcal{C}$ is exactly the set of atoms that generates $\mathcal{F}$ and that it is either finite or countable. By part b), a union of any subcollection of $\mathcal{C}$ produces a distinct subset of $\mathcal{F}$. Thus, if $\mathcal{C}$ is finite, it's power set is also finite. If $\mathcal{C}$ is countable, its power set is uncountable (See PSET 1, exercise 1). Either way, this contradicts the original assumption.

Problem 5: Random variables

Let $\mathcal{F}$ be a σ-field on a set Ω and X_1, X_2 be two $\mathcal{F}$-measurable random variables taking a finite number of values in $\mathbb{R}$. Let also $Y = X_1 + X_2$. From the course, we know that it always holds that $\sigma(Y) \subset \sigma(X_1, X_2)$, i.e., that X_1, X_2 carry together at least as much information as Y, but that the reciprocal statement is not necessarily true.

a) Provide a non-trivial example of random variables X_1, X_2 such that $\sigma(Y) = \sigma(X_1, X_2)$.

b) Provide a non-trivial example of random variables X_1, X_2 such that $\sigma(Y) \neq \sigma(X_1, X_2)$.

c) Assume that there exists $\omega_1 \neq \omega_2$ and $a \neq b$ such that $X_1(\omega_1) = X_2(\omega_2) = a$ and $X_1(\omega_2) = X_2(\omega_1) = b$. Is it possible in this case that $\sigma(Y) = \sigma(X_1, X_2)$?

Solution

a) Consider e.g. X_1 taking values in $\{0, 1\}$ and X_2 taking values in $\{0, 2\}$. Then it is possible to deduce the values of both X_1 and X_2 from the sole value of Y, so $\sigma(Y) = \sigma(X_1, X_2)$ (as an exercise, write this down formally).

b) Consider e.g. X_1 taking values in $\{3, 5\}$ and X_2 taking values in $\{7, 9\}$. When $Y(\omega) = 12$, it is impossible to tell whether $X_1(\omega) = 3, X_2(\omega) = 9$ or $X_1(\omega) = 5, X_2(\omega) = 7$. The random variable Y carries then less information than the two random variables X_1, X_2 together (again, as an exercise, write this down formally).

c) The answer is no, i.e., $\sigma(Y) \neq \sigma(X_1, X_2)$, as when $Y(\omega) = a + b$, we will not be able to tell whether $\omega = \omega_1$ or $\omega = \omega_2$.