Problem Set 5

Problem 1: Optimal coupling

Let X and Y be random variables on common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The total variation distance is defined as

$$d_{TV}(X,Y) = \sup_{A \in \mathcal{B}(\mathbb{R})} |(\mathbb{P}(\{X \in A\}) - \mathbb{P}(\{Y \in A\}))|.$$

a) Show that

$$\mathbb{P}(\{X = Y\}) \leq 1 - d_{TV}(X,Y).$$

b) Let X and Y be discrete and supported on $\{0, 1, 2, \dots\}$ with pmfs p_X and p_Y. Show that

$$d_{TV}(X,Y) = \frac{1}{2} \sum_{m \geq 0} |p_X(m) - p_Y(m)|.$$

c) (Optimal coupling) Let $\Omega = \{1, \dots k\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, and $\mathbb{P}(\omega) = \frac{1}{k} \quad \forall \omega \in \Omega$. Suppose that X is Bernoulli $\left(\frac{k-1}{k}\right)$ and Y is Bernoulli $\left(\frac{2}{k}\right)$. What is $d_{TV}(X,Y)$?

Construct an explicit mappings $X \colon \Omega \to \{0,1\}$ and $Y \colon \Omega \to \{0,1\}$ so that the bound in part a) is satisfied with equality.

Problem 2: Poisson coupling

a) Show that a sum of two independent Poisson random variables is a Poisson random variable.

b) Let X be a Poisson random variable with parameter λ and Y a Poisson random variable with parameter ν, $\lambda > \nu$. Use the coupling inequality to show that

$$d_{TV}(X,Y) \leq \lambda - \nu.$$

Hint: Use part a) to construct a coupling of X and Y.