
Problem Set 1

Review. Recall that if there exists a 1-1 mapping of A onto B , we say that A and B have the same *cardinal number*, or, that A and B are equivalent, and we write $A \sim B$.

Definition. For any positive integer n , let $J_n = \{1, 2, \dots, n\}$ be the set of first n positive integers and J be the set of all positive integers. For any A we say

- (a) A is *finite* if $A \sim J_n$ for some n or $A = \emptyset$.
- (b) A is *infinite* if A is not finite.
- (c) A is *countable* if $A \sim J$.
- (d) A is *uncountable* if A is neither finite nor countable.
- (e) A is *at most countable* if A is either finite or countable.

Problem 1: Size of Infinity

Let \mathbb{Q} denote rational numbers and \mathbb{R} denote real numbers.

- a) Show that \mathbb{Q} is countable.
- b) Show that every infinite subset of a countable set A is countable.

Remark: Roughly speaking, countable sets represent the “smallest” infinity.

- c) Let the set A be all sequences whose elements are the digits 0 and 1. Show that A is not countable. Conclude that \mathbb{R} is not countable.
- d) Are irrational numbers, e.g. $\mathbb{R} \setminus \mathbb{Q}$, countable? Why or why not?
- e) Construct a set that is infinite, but does not have the same cardinal number as \mathbb{Q} or \mathbb{R} .

Problem 2: Atoms

Let $\Omega = \{1, \dots, 6\}$ and $\mathcal{A} = \{\{1, 2, 3\}, \{1, 3, 5\}\}$.

- a) Describe $\mathcal{F} = \sigma(\mathcal{A})$, the σ -field generated by \mathcal{A} .

Hint: For a finite set Ω , the number of elements of a σ -field on Ω is always a power of 2.

- b) Give the list of non-empty elements G of \mathcal{F} such that

$$\text{if } F \in \mathcal{F} \text{ and } F \subset G, \text{ then } F = \emptyset \text{ or } G.$$

These elements are called the *atoms* of the σ -field \mathcal{F} (cf. course). Equivalently, an event $G \in \mathcal{F}$ is *not* an atom if there exists $F \in \mathcal{F}$ such that $F \neq \emptyset$, $F \subset G$ and $F \neq G$.

The atoms of a \mathcal{F} form a *partition* of the set Ω and they also generate the σ -field \mathcal{F} in this case. (note also that if m is the number of atoms of \mathcal{F} , then the number of elements of \mathcal{F} equals 2^m)

c) Let $X_1(\omega) = 1_{\{1,2,3\}}(\omega)$, $X_2 = 1_{\{1,3,5\}}(\omega)$ and $Y(\omega) = X_1(\omega) + X_2(\omega)$. Does it hold that $\sigma(Y) = \sigma(X_1, X_2)$?

Problem 3: Borel atoms

Let now $\Omega = [0, 1]$ and $\mathcal{F} = \mathcal{B}([0, 1])$ be the Borel σ -field on $[0, 1]$.

- What are the atoms of \mathcal{F} ?
- Is it true in this case that the σ -field \mathcal{F} is generated by its atoms?
- Describe the σ -field $\sigma(\{x\}, x \in [0, 1])$.

Problem 4: Sigma-field generated by atoms

Let Ω be an arbitrary set and \mathcal{F} be a σ -field on Ω . In this problem we will show that if \mathcal{F} is infinite, it must be uncountable. We will proceed with proof by contradiction and assume that \mathcal{F} is countable.

- For every $\omega \in \Omega$, define $B_\omega = \bigcap_{A \in \mathcal{F}: \omega \in A} A$. Is $B_\omega \in \mathcal{F}$? Why or why not?
- Let $\mathcal{C} = \{B_\omega\}_{\omega \in \Omega}$ be a collection of all such unique B_ω . Argue that \mathcal{C} partitions Ω and that it is at most finite, or countable.
- Argue that $\sigma(\mathcal{C}) = \mathcal{F}$. That is, the σ -field generated by \mathcal{C} is exactly \mathcal{F} .
- Conclude from parts (a) - (c) that there is a contradiction and it is not possible for \mathcal{F} to be countable.

Problem 5: Random variables

Let \mathcal{F} be a σ -field on a set Ω and X_1, X_2 be two \mathcal{F} -measurable random variables taking a finite number of values in \mathbb{R} . Let also $Y = X_1 + X_2$. From the course, we know that it always holds that $\sigma(Y) \subset \sigma(X_1, X_2)$, i.e., that X_1, X_2 carry together at least as much information as Y , but that the reciprocal statement is not necessarily true.

- Provide a non-trivial example of random variables X_1, X_2 such that $\sigma(Y) = \sigma(X_1, X_2)$.
- Provide a non-trivial example of random variables X_1, X_2 such that $\sigma(Y) \neq \sigma(X_1, X_2)$.
- Assume that there exists $\omega_1 \neq \omega_2$ and $a \neq b$ such that $X_1(\omega_1) = X_2(\omega_2) = a$ and $X_1(\omega_2) = X_2(\omega_1) = b$. Is it possible in this case that $\sigma(Y) = \sigma(X_1, X_2)$?