Advanced Probability & Applications
Prof. Yanina Shkel

Ecole Polytechnique Fédérale, Lausanne
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Problem Set 11

Problem 1: Properties

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space, X be an integrable random variable defined on this space and let $\mathcal{G}$ be a sub-σ-field of $\mathbb{F}$. Relying only on the definition of conditional expectation, show the following properties:

a) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$.

b) If X is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

c) If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$ a.s.

d) If Y is $\mathcal{G}$-measurable and bounded, then $\mathbb{E}(XY|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})\, Y$ a.s.

e) If $\mathcal{H}$ is a sub-σ-field of $\mathcal{G}$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) = \mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H})$ a.s.

Hint for parts b) to e): According to the course definition, in order to check that some candidate random variable Z is the conditional expectation of X given $\mathcal{G}$, you should check the following two conditions:

(i) Z is $\mathcal{G}$-measurable;

(ii) Z satisfies $\mathbb{E}((Z - X)\, U) = 0$ for every U $\mathcal{G}$-measurable and bounded.

Problem 2: Discrete conditional expectation

Let X, Y be two discrete random variables (with values in a countable set C). Let us moreover assume that X is integrable.

a) Show that the random variable $\psi(Y)$, where ψ is defined as

$$\psi(y) = \sum_{x \in C} x\, \mathbb{P}(\{X = x\}|\{Y = y\})$$

matches the definition of conditional expectation $\mathbb{E}(X|Y)$ given in the lectures.

b) *Application:* One rolls two independent and balanced dice (say Y and Z), each with four faces. What is the conditional expectation of the maximum of the two, given the value of one of them?

Problem 3: Zombie apocalypse

A zombie bites N people during its lifetime (un-death time?), where N has the Poisson distribution with parameter λ. Once a person is bitten, they become a zombie with probability p or their brain is eaten with probability $q = 1 - p$, independently of all other bitten individuals. Let Z be the number of zombie offsprings created by a zombie.

a) Find $\mathbb{E}(Z|N)$ and $\mathbb{E}(Z)$

b) Find $\mathbb{E}(N|Z)$

c) *Optional*: Assume that the initial zombie infestation is small (relative to the population of the planet). According to the above model, for which values of p and λ would you expect the infestation to turn into a full blown apocalypse and for which values would it die down locally?