

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 6
Homework 3

Information Theory and Coding
Sep. 22, 2025

PROBLEM 1. Recall that for a code $\mathcal{C} : \mathcal{U} \rightarrow \{0, 1\}^*$, we define $\mathcal{C}^n : \mathcal{U}^n \rightarrow \{0, 1\}^*$ as $\mathcal{C}^n(u_1 \dots u_n) = \mathcal{C}(u_1) \dots \mathcal{C}(u_n)$.

- (a) Show that if \mathcal{C} is uniquely decodable, then for all $n \geq 1$, \mathcal{C}^n is injective.
- (b) Suppose \mathcal{C} is not uniquely decodable. Show that there are u^n and v^m such that $u_1 \neq v_1$ and $\mathcal{C}^n(u^n) = \mathcal{C}^m(v^m)$.
- (c) Suppose \mathcal{C} is not uniquely decodable. Show that there is a k such that \mathcal{C}^k is not injective. [Hint: try $k = n + m$.]

PROBLEM 2. Suppose X, Y and Z are random variables.

- (a) Show that $H(X) + H(Y) + H(Z) \geq \frac{1}{2}[H(XY) + H(YZ) + H(ZX)]$.
- (b) Show that $H(XY) + H(YZ) \geq H(XYZ) + H(Y)$.
- (c) Show that

$$2[H(XY) + H(YZ) + H(ZX)] \geq 3H(XYZ) + H(X) + H(Y) + H(Z).$$

- (d) Show that $H(XY) + H(YZ) + H(ZX) \geq 2H(XYZ)$.
- (e) Suppose n points in three dimensions are arranged so that their their projections to the xy , yz and zx planes give n_{xy} , n_{yz} and n_{zx} points. Clearly $n_{xy} \leq n$, $n_{yz} \leq n$, $n_{zx} \leq n$. Use part (d) show that

$$n_{xy}n_{yz}n_{zx} \geq n^2.$$

PROBLEM 3. Let X be a random variable taking values in M points a_1, \dots, a_M , and let $P_X(a_M) = \alpha$. Show that

$$H(X) = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha)H(Y)$$

where Y is a random variable taking values in $M - 1$ points a_1, \dots, a_{M-1} with probabilities $P_Y(a_j) = P_X(a_j)/(1 - \alpha)$; $1 \leq j \leq M - 1$. Show that

$$H(X) \leq \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha} + (1 - \alpha) \log(M - 1)$$

and determine the condition for equality.

PROBLEM 4. Let X, Y, Z be discrete random variables. Prove the validity of the following inequalities and find the conditions for equality:

- (a) $I(X, Y; Z) \geq I(X; Z)$.
- (b) $H(X, Y|Z) \geq H(X|Z)$.

(c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

(d) $I(X; Z|Y) \geq I(Z; Y|X) - I(Z; Y) + I(X; Z)$.

PROBLEM 5. For a stationary process X_1, X_2, \dots , show that

(a) $\frac{1}{n}H(X_1, \dots, X_n) \geq H(X_n|X_{n-1}, \dots, X_1)$.

(b) $\frac{1}{n}H(X_1, \dots, X_n) \leq \frac{1}{n-1}H(X_1, \dots, X_{n-1})$.

PROBLEM 6. Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove that

$$H(X_0|X_{-1}, \dots, X_{-n}) = H(X_0|X_1, \dots, X_n).$$

That is: the conditional entropy of the present given the past is equal to the conditional entropy of the present given the future.

PROBLEM 7. Let $X \leftrightarrow Y \leftrightarrow (Z, W)$ form a Markov chain. Show that

$$I(X; Z) + I(X; W) \leq I(X; Y) + I(Z; W)$$