

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

School of Computer and Communication Sciences

Handout 23
Homework 10

Information Theory and Coding
Nov. 24, 2025

PROBLEM 1. Suppose U is $\{0, 1\}$ valued with $\mathbb{P}(U = 0) = \mathbb{P}(U = 1) = 1/2$. Suppose we have a distortion measure d given by

$$d(u, v) = \begin{cases} 0, & u = v \\ 1, & (u, v) = (1, 0) \\ \infty, & (u, v) = (0, 1) \end{cases}$$

i.e., we never want to represent a 0 with a 1. Find $R(D)$.

PROBLEM 2. Suppose $\mathcal{U} = \mathcal{V}$ are additive groups with group operation \oplus . (E.g., $\mathcal{U} = \mathcal{V} = \{0, \dots, K-1\}$, with modulo K addition.) Suppose the distortion measure $d(u, v)$ depends only on the difference between u and v and is given by $g(u \ominus v)$. Let $\phi(D)$ denote $\max\{H(Z) : E[g(Z)] \leq D\}$.

a) Show that $\phi(D)$ is concave.

b) Let (U, V) be such that $E[d(U, V)] \leq D$. Show that $I(U; V) \geq H(U) - \phi(D)$ by justifying

$$I(U; V) = H(U) - H(U|V) = H(U) - H(U \ominus V|V) \geq H(U) - H(U \ominus V) \geq H(U) - \phi(D).$$

c) Show that $R(D) \geq H(U) - \phi(D)$.

d) Assume now that U is uniform on \mathcal{U} . Show that $R(D) = H(U) - \phi(D)$.

PROBLEM 3. Suppose $\mathcal{U} = \mathcal{V} = \mathbb{R}$, the set of real numbers, and $d(u, v) = (u - v)^2$.

(a) Show that for any U with variance σ^2 , $R(D)$ satisfies

$$h(U) - \frac{1}{2} \log(2\pi e D) \leq R(D).$$

(b) Show that $R(D)$ does not depend on the mean of U .

Now, assume without loss of generality that U is zero-mean. Suppose we have access to a noisy observation V of U through the channel $U + Z = V$, where $Z \sim \mathcal{N}(0, \sigma_Z^2)$ and independent of U . We reconstruct U via a linear estimator $\hat{U} = aV + b$.

(c) Show that $a = \frac{\sigma^2}{\sigma^2 + \sigma_Z^2}$ and $b = 0$ minimizes $E[(U - \hat{U})^2]$ and for such choice of a, b ,

$$E[(U - \hat{U})^2] = \sigma^2 \frac{\sigma_Z^2}{\sigma^2 + \sigma_Z^2}.$$

(d) For the channel above, show that

$$I(U; V) \leq \frac{1}{2} \log\left(1 + \frac{\sigma^2}{\sigma_Z^2}\right)$$

(e) Show that for $D \leq \sigma^2$

$$R(D) \leq \frac{1}{2} \log(\sigma^2/D).$$

[Hint: Use the channel above for a candidate $p_{V|U}$.]

PROBLEM 4. Given finite alphabets \mathcal{X} and \mathcal{Y} , a distribution p_{XY} , $0 < \epsilon < \epsilon'$, and a sequence $x^n \in T(n, p_X, \epsilon)$, consider a random vector Y^n with independent components with $\Pr(Y_i = y) = p_{Y|X}(y|x_i)$.

For $x \in \mathcal{X}$, let $J(x) = \{i : x_i = x\}$. For an $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, let $N(x, y) = \sum_i \mathbb{1}\{(x_i, Y_i) = (x, y)\} = \sum_{i \in J(x)} \mathbb{1}\{Y_i = y\}$.

- (a) Show that for each x and y , $np(x, y)(1 - \epsilon) \leq E[N(x, y)] \leq np(x, y)(1 + \epsilon)$, and $\text{Var}(N(x, y))$ is at most n . [Hint: don't forget that x^n is in $T(n, p_X, \epsilon)$.]
- (b) Show that for each x and y , both $\Pr(N(x, y) < np(x, y)(1 - \epsilon'))$ and $\Pr(N(x, y) > np(x, y)(1 + \epsilon'))$ approach to zero as n gets large. Would this be true if we had not assumed $\epsilon < \epsilon'$?
- (c) Using (a) and (b) show that $\Pr((x^n, Y^n) \notin T(n, p_{XY}, \epsilon'))$ approaches 0 as n gets large.
- (d) Suppose now we have a distribution $p(u, x, y)$ where $p(y|ux) = p(y|x)$. [In other words, U, X, Y form a Markov chain.] Suppose (u^n, x^n) is in $T(n, p_{UX}, \epsilon)$, and Y^n has independent components as above. What can we say about $\Pr((u^n, x^n, Y^n) \in T(n, p_{UXY}, \epsilon'))$?