Solution 8
Introduction to Quantum Information Processing

Exercise 1 *Decoherence model*

(a) The global state of the system is:

$$|\psi_0\rangle = |\mathcal{E}\rangle \otimes |\phi_0\rangle$$

(b) First of all:

$$U^\dagger = \left(\sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i)\right)^\dagger = \sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i)^\dagger$$

Thus because $(|i\rangle)$ is an orthonormal basis:

$$U^\dagger U = \sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i)^\dagger \mathcal{D}(\theta_i)$$

Now it is easy to show that $\mathcal{D}(\theta_i)^\dagger \mathcal{D}(\theta_i) = I$ so that $U^\dagger U = I$

(c) We have:

$$\mathcal{D}(\theta_i)^n = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_i} \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ 0 & e^{ni\theta_i} \end{pmatrix} = \mathcal{D}(n\theta_i)$$

Thus because $(|i\rangle)$ is an orthonormal basis:

$$U^n = \left(\sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i)\right)^n = \sum_{i=1}^{\infty} |i\rangle \langle i| \otimes (\mathcal{D}(\theta_i))^n = \sum_{i=1}^{\infty} |i\rangle \langle i| \otimes \mathcal{D}(n\theta_i)$$

Finally, because $\mathcal{P}(|\mathcal{E}\rangle \to |i\rangle) = |\langle i|\mathcal{E}\rangle|^2 = \mu(\theta_i)^2$ then $\langle i|\mathcal{E}\rangle = \sqrt{\mu(\theta_i)} e^{i \arg\langle i|\mathcal{E}\rangle}$ and thus we have:

$$|\psi_n\rangle = \sum_{i=1}^{\infty} |i\rangle \langle i|\mathcal{E}\rangle \otimes \mathcal{D}(n\theta_i) |\phi_0\rangle = \sum_{i=1}^{\infty} e^{i \arg\langle i|\mathcal{E}\rangle} \sqrt{\mu(\theta_i)}\, |i\rangle \otimes \mathcal{D}(n\theta_i) |\phi_0\rangle$$

(d) Using question (a) we find:

$$\rho_0 = |\phi_0\rangle \langle\phi_0| = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

The Von Neumann entropy is $S_0 = 0$ as this is a rank-one matrix with one eigenvalue (=1).

(e) We find:

$$\mathcal{D}(\theta) |\phi_0\rangle = \begin{pmatrix} \alpha \\ e^{i\theta}\beta \end{pmatrix}$$

Thus:

$$\mathcal{D}(\theta)\rho_0\mathcal{D}(\theta)^\dagger = \begin{pmatrix} \alpha \\ e^{i\theta}\beta \end{pmatrix} \begin{pmatrix} \alpha^* & e^{-i\theta}\beta^* \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & e^{-i\theta}\alpha\beta^* \\ e^{i\theta}\alpha^*\beta & |\beta|^2 \end{pmatrix}$$

(f) Using question (c) we have:

$$|\psi_n\rangle\langle\psi_n| = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} e^{i\arg\langle\mathcal{E}|\theta_i|\mathcal{E}|\theta_i\rangle - i\arg\langle\mathcal{E}|\theta_j|\mathcal{E}|\theta_j\rangle}\sqrt{\mu(\theta_i)\mu(\theta_j)}\,|j\rangle\langle i| \otimes \mathcal{D}(n\theta_j)\,|\phi_0\rangle\langle\phi_0|\,\mathcal{D}(n\theta_i)^\dagger$$

Therefore, using (e):

$$\rho_n = \sum_{i=1}^{\infty}\mu(\theta_i)\mathcal{D}(n\theta_i)\rho_0\mathcal{D}(n\theta_i)^\dagger \tag{1}$$

$$= \begin{pmatrix} \sum_{i=1}^{\infty}\mu(\theta_i)|\alpha|^2 & \sum_{i=1}^{\infty}\mu(\theta_i)\alpha\beta^* e^{-in\theta} \\ \sum_{i=1}^{\infty}\mu(\theta_i)\alpha\beta^* e^{in\theta} & \sum_{i=1}^{\infty}\mu(\theta_i)|\beta|^2 \end{pmatrix} \tag{2}$$

$$= \begin{pmatrix} \mathbb{E}_{\hat\theta}[|\alpha|^2] & \mathbb{E}_{\hat\theta}[\alpha\beta^* e^{-in\hat\theta}] \\ \mathbb{E}_{\hat\theta}[\alpha\beta^* e^{in\hat\theta}] & \mathbb{E}_{\hat\theta}[|\beta|^2] \end{pmatrix} \tag{3}$$

(g) This is a direct application of the MGF of $\hat\theta$. The limit is thus:

$$\rho_\infty = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \tag{4}$$

(h) The entropy initially is $S_0 = 0$ since ρ_0 is in fact pure (or a rank one matrix) and increases to attain its maximum $S_\infty = -|\alpha|^2 \ln|\alpha|^2 - |\beta|^2 \ln|\beta|^2$.