Solution 7
Introduction to Quantum Information Processing

Exercise 1 *Partial density matrices: illustrative examples*

a) The corresponding density matrix is

$$\rho_{AB} = |\Psi\rangle\langle\Psi| = \epsilon^2|00\rangle\langle 00| + \epsilon\sqrt{1-\epsilon^2}\,(|00\rangle\langle 11| + |11\rangle\langle 00|) + (1-\epsilon^2)|11\rangle\langle 11|.$$

We compute the reduced density matrix of subsystem A by tracing out subsystem B:

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}).$$

Using $\mathrm{Tr}_B(|ij\rangle\langle kl|) = \langle l|j\rangle\,|i\rangle\langle k|$, we get:

$$\rho_A = \epsilon^2|0\rangle\langle 0| + (1-\epsilon^2)|1\rangle\langle 1|.$$

Similarly,

$$\rho_B = \epsilon^2|0\rangle\langle 0| + (1-\epsilon^2)|1\rangle\langle 1|.$$

Hence, both subsystems have identical reduced states:

$$\rho_A = \rho_B = \begin{pmatrix} \epsilon^2 & 0 \\ 0 & 1-\epsilon^2 \end{pmatrix}.$$

In Bloch-vector form, any single-qubit density matrix can be written as $\rho = \frac{1}{2}\big(I + \vec{r}\cdot\vec{\sigma}\big)$, so here

$$r_x = 0, \quad r_y = 0, \quad r_z = 2\epsilon^2 - 1.$$

Thus, the Bloch vector is $\vec{r} = (0, 0, 2\epsilon^2 - 1)$.

- For $\epsilon = 1$, $|\Psi\rangle = |00\rangle$, which is a separable state. Then $\rho_A = |0\rangle\langle 0|$ corresponds to the north pole of the Bloch sphere ($r_z = +1$).
- For $\epsilon = 0$, $|\Psi\rangle = |11\rangle$, also separable. Then $\rho_A = |1\rangle\langle 1|$ corresponds to the south pole ($r_z = -1$).
- For $\epsilon = \frac{1}{\sqrt{2}}$, the state is maximally entangled (a Bell state), and $\rho_A = \frac{1}{2}I$. The Bloch vector is $\vec{r} = 0$, i.e. the center of the Bloch ball.

As ϵ varies from 0 to 1, the Bloch vector moves along the z-axis from the south pole to the north pole, passing through the center at $\epsilon = 1/\sqrt{2}$.

b) We can group the terms according to A and B:

$$|\Phi\rangle_{AB} = \frac{1}{2}\big(|0\rangle_A(|0\rangle_B + |1\rangle_B) + |1\rangle_A(|0\rangle_B + |1\rangle_B)\big) = \frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A) \otimes \frac{1}{\sqrt{2}}(|0\rangle_B + |1\rangle_B).$$

Hence the state factorizes:

$$|\Phi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B, \quad \text{with} \quad |\psi\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Therefore,

$$\rho_A = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \rho_B = \rho_A.$$

Rank of the partial density matrices:
Since $\rho_A = |\psi\rangle\langle\psi|$ is a pure state, its rank is 1. The same holds for ρ_B.

Representation on the Bloch ball:
We can write ρ_A in Bloch form:

$$\rho_A = \frac{1}{2}\big(I + \vec{r}\cdot\vec{\sigma}\big) \quad \Rightarrow \quad \vec{r} = (1, 0, 0).$$

Thus, the Bloch vector lies on the equator of the Bloch sphere along the $+x$-axis.

c) We can group terms by subsystem A:

$$|\Phi'\rangle_{AB} = \frac{1}{2}\big[|0\rangle_A(|0\rangle_B + |1\rangle_B) + |1\rangle_A(-|0\rangle_B + |1\rangle_B)\big] = \frac{1}{2}\big[|0\rangle_A|+\rangle_B - |1\rangle_A|-\rangle_B\big]$$

This is a maximally entangled state. The total density matrix is $\rho_{AB} = |\Phi'\rangle\langle\Phi'|$. Tracing out subsystem B, we get

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \frac{1}{2}\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big) = \frac{1}{2}I.$$

By symmetry,

$$\rho_B = \frac{1}{2}I.$$

Rank of the reduced density matrices:

$$\mathrm{rank}(\rho_A) = \mathrm{rank}(\rho_B) = 2.$$

They are both mixed states.

Explanation: The presence of the minus sign makes the state entangled. In the previous case, the state was separable and the reduced density matrices were pure (rank 1). Now, because $|\Phi'\rangle$ is maximally entangled, tracing out one subsystem yields a completely mixed state of rank 2.

Representation on the Bloch ball:

$$\rho_A = \frac{1}{2}I \quad \Rightarrow \quad \vec{r} = (0, 0, 0).$$

The Bloch vector lies at the center of the Bloch sphere.

d) The density matrix of the state is:

$$\rho_{ABC} = |W\rangle\langle W| = \frac{1}{3}\Big(|001\rangle\langle 001| + |001\rangle\langle 010| + |001\rangle\langle 100| + |010\rangle\langle 001| + |010\rangle\langle 010| + |010\rangle\langle 100| + |100\rangle\langle 001| + |100\rangle\langle 010| + |100\rangle\langle 100|\Big)$$

Hence,

$$\rho_A = \mathrm{Tr}_{BC}(\rho_{ABC}) = \frac{1}{3}\Big(2|0\rangle\langle 0| + |1\rangle\langle 1|\Big) = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}.$$

$$\rho_{BC} = \mathrm{Tr}_A(\rho_{ABC}) = \frac{1}{3}\Big((|01\rangle + |10\rangle)(\langle 01| + \langle 10|) + |00\rangle\langle 00|\Big).$$

Expanding in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$,

$$\rho_{BC} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Because the $|W\rangle$ state is symmetric under permutation of the qubits, the reduced states for $B|AC$ and $C|AB$ are identical:

$$\rho_B = \rho_C = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \qquad \rho_{AC} = \rho_{AB} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Exercise 2 *Tableau computation of partial density matrices*

We consider a bipartite 2×2 system, with the total density matrix

$$M = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix},$$

whose rows and columns are ordered according to $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.
The partial trace over the first subsystem corresponds to

$$M^{(1)}_{jl} = \sum_{m=0,1} M_{mj;ml}.$$

This means we sum the diagonal 2×2 blocks of the matrix M:

$$M = \begin{pmatrix} \boxed{\begin{matrix} a & b \\ e & f \end{matrix}} & \begin{matrix} c & d \\ g & h \end{matrix} \\ \begin{matrix} i & j \\ m & n \end{matrix} & \boxed{\begin{matrix} k & l \\ o & p \end{matrix}} \end{pmatrix}.$$

Taking the sum of the two boxed blocks gives

$$M^{(1)} = \begin{pmatrix} a+k & b+l \\ e+o & f+p \end{pmatrix}.$$

Thus, the partial trace over the first subsystem simply adds the diagonal 2×2 blocks of the 4×4 matrix.

The partial trace over the second subsystem corresponds to

$$M^{(1)}_{ik} = \sum_{m=0,1} M_{im;km}.$$

This means we sum the 2×2 blocks elementwise along the diagonal inside each block:

$$M^{(1)} = \begin{pmatrix} a+f & c+h \\ i+n & k+p \end{pmatrix}.$$

Thus, tracing over the second subsystem amounts to summing the diagonal entries of each 2×2 block in the tableau representation.

This provides a convenient way to compute partial traces directly from the 4×4 array representation without reverting to Dirac notation.

Exercise 3 *The difference between a Bell state and a statistical mixture of* $|00\rangle$, $|11\rangle$

a) For the Bell state the density matrix is simply

$$\rho_{\text{Bell}} = |B_{00}\rangle\langle B_{00}| = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

In array form

$$\rho_{\text{Bell}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Note this is a rank one matrix as it should since ρ_{Bell} is a rank one projector with eigenvalues 1 and $0, 0, 0$. Note also that teh criterion for a pure state is satisfied: $\rho^2_{\text{Bell}} = \rho_{\text{Bell}}$. Sanity check $Tr\rho_{\text{Bell}} = 1$.

b) For the statistical mixture we have

$$\rho_{\text{stat}} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$$

In array form

$$\rho_{\text{stat}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note this is a rank two matrix with eigenvalues $1, 0, 0, 1$. Note also that $\rho^2_{\text{stat}} \neq \rho_{\text{stat}}$. Sanity check: $Tr\rho_{\text{stat}} = 1$.

c) In the Bell state the average of the observable $\mathcal{B}$ is

$$Tr(\mathcal{B}\rho_{\text{Bell}}) = Tr(\mathcal{B}|B_{00}\rangle\langle B_{00}|) = Tr\langle B_{00}|\mathcal{B}|B_{00}\rangle = \langle B_{00}|\mathcal{B}|B_{00}\rangle$$

The expression as a function of angles is calulated in the course

$$\cos 2(\alpha - \beta) + \cos 2(\alpha - \beta') - \cos 2(\alpha' - \beta) + \cos 2(\alpha' - \beta')$$

and for the optimal choice of angles the values is $2\sqrt{2}$.
In the statistical state we have by linearity and cyclicity of the trace

$$Tr(\mathcal{B}\rho_{\text{stat}}) = \frac{1}{2}\langle 00|\mathcal{B}|00\rangle + \frac{1}{2}\langle 11|\mathcal{B}|11\rangle$$

For $A \otimes B$ we get the contribution

$$\frac{1}{2}\langle 0|A|0\rangle\langle 0|B|0\rangle + \frac{1}{2}\langle 1|A|1\rangle\langle 1|B|1\rangle = (\cos^2\alpha - \sin^2\alpha)(\cos^2\beta - \sin^2\beta) = \cos 2\alpha \cos 2\beta$$

So for the correlation coefficient we have

$$Tr(\mathcal{B}\rho_{\text{stat}}) = \cos 2\alpha \cos 2\beta + \cos 2\alpha \cos 2\beta' - \cos 2\alpha' \cos 2\beta + \cos 2\alpha' \cos 2\beta'$$

For the optimal angles of CSHS we find $\sqrt{2}$. Note that it is possible to prove this expression can never be greater than 2.

Exercise 4 *Dynamics of one-qubit density matrix*

(a) From Homework 5 we can write down with $\alpha_t^2 + \beta_t^2 = 1$ and $n_x^2 + n_z^2 = 1$:

$$U_t = \alpha I + i\beta\,(n_x\sigma_x + n_z\sigma_z) \tag{1}$$

where $\alpha := \cos\left(\frac{\sqrt{\omega_1^1+\delta^2}}{2}t\right)$, $\beta := \sin\left(\frac{\sqrt{\omega_1^1+\delta^2}}{2}t\right)$, $n_x := \frac{\omega_1}{\sqrt{\omega_1^2+\delta^2}}$, $n_z := -\frac{\delta}{\sqrt{\omega_1^2+\delta^2}}$.
We also know that the time-evolved state is described by

$$\rho(t) = U_t\rho(0)U_t^\dagger.$$

This is a long computation for which we do not show all the details. To lighten the notation, we write $a_i(0) = a_i$ here. We first compute the part $\rho(0)U_t^\dagger$:

$$\begin{aligned}
\rho(0)U_t^\dagger &= \left[\frac{1}{2}(\mathbb{I} + \vec{a}\cdot\vec{\sigma})\right]U_t^\dagger \\
&= \frac{1}{2}\left[U_t^\dagger + \alpha\vec{a}\cdot\vec{\sigma} + i\beta(a_x\sigma_x + a_y\sigma_y + a_z\sigma_z)(n_x\sigma_x - n_z\sigma_z)\right] \\
&= \frac{1}{2}\Big[U_t^\dagger + i\beta(a_xn_x - a_zn_z)\mathbb{I} + \Big(\alpha a_x + n_z\beta a_y\Big)\sigma_x + \\
&\quad + \Big(\alpha a_y - n_z\beta a_x + n_x\beta a_z\Big)\sigma_y + \Big(\alpha a_z - n_x\beta a_y\Big)\sigma_z\Big].
\end{aligned}$$

What is left to do now is to compute $U_t(\rho(0)U_t^\dagger)$.

$$\begin{aligned}
U_t\rho(0)U_t^\dagger &= U_t \times \frac{1}{2}\Big[U_t^\dagger + i\beta(a_x n_x - a_z n_z)\mathbb{I} + \Big(\alpha a_x n_z \beta a_y\Big)\sigma_x + \\
&\quad + \Big(\alpha a_y - n_z\beta a_x + n_x\beta a_z\Big)\sigma_y + \Big(\alpha a_z - n_x\beta a_y\Big)\sigma_z\Big] \\
&= \frac{1}{2}\Big\{\mathbb{I} + \Big[a_x(0)\Big(\alpha^2 + \beta^2 n_x^2 - \beta^2 n_z^2\Big) + 2a_y(0)\alpha\beta n_z + 2a_z(0)\beta^2 n_x n_z\Big]\sigma_x + \\
&\quad + \Big[-2a_x(0)\alpha\beta n_z + a_y(0)\Big(\alpha^2 - \beta^2\Big) + 2\alpha\beta a_z(0)n_x\Big]\sigma_y + \\
&\quad + \Big[2a_x(0)n_x n_z\beta^2 - 2a_y(0)n_x\beta\alpha + a_z(0)\Big(\alpha^2 - n_x^2\beta^2 + n_z^2\beta^2\Big)\Big]\sigma_z\Big\} \\
&= \frac{1}{2}\Big\{\mathbb{I} + a_x(t)\sigma_x + a_y(t)\sigma_y + a_z(t)\sigma_z\Big\}
\end{aligned}$$

where we defined the time-evolved coefficients

$$\begin{aligned}
a_x(t) &:= a_x(0)\Big(\alpha^2 + \beta^2 n_x^2 - \beta^2 n_z^2\Big) + 2a_y(0)\alpha\beta n_z + 2a_z(0)\beta^2 n_x n_z \\
a_y(t) &:= -2a_x(0)\alpha\beta n_z + a_y(0)\Big(\alpha^2 - \beta^2\Big) + 2a_z(0)\alpha\beta n_x \\
a_z(t) &:= 2a_x(0)n_x n_z\beta^2 - 2a_y(0)n_x\beta\alpha + a_z(0)\Big(\alpha^2 - n_x^2\beta^2 + n_z^2\beta^2\Big)
\end{aligned}$$

(b) One can check this after a long calculation using the previous formulas

(c) It suffices to notice that $1 - \|a_t\|^2 = \det(\rho_t) = \det(\rho) = 1 - \|a\|^2$