
Homework 8
Introduction to Quantum Information Processing

Exercise 1 *Density matrix: a decoherence model*

In the following, we will study a model of decoherence of one qubit interacting with the environment. The whole system is defined in the hilbert space $\mathcal{H} = \mathcal{H}_{\mathcal{E}} \otimes \mathcal{H}_b$ where $\mathcal{H}_{\mathcal{E}}$ is the Hilbert space describing the possible states of the environment and $\mathcal{H}_b = \mathbb{C}^2$ is the Hilbert space describing the possible states of the qubit.

Let $|\phi_0\rangle = \alpha |0\rangle + \beta |1\rangle \in \mathcal{H}_b$ be the initial state of the qubit and $|\mathcal{E}\rangle \in \mathcal{H}_b$ that of the environment (or sometimes called *heat-bath*). Let $(|i\rangle)_{i\geq 1} \in \mathcal{H}_{\mathcal{E}}$ be an "infinite" orthonormal basis of the environment $\mathcal{H}_{\mathcal{E}}$. We define the evolution operator $U = \sum_{i=1}^{+\infty} |i\rangle \langle i| \otimes \mathcal{D}(\theta_i)$ for some distinct angles $\theta_i \in \mathbb{R}$, and the dephasing operator: $\mathcal{D}(\theta_i) = |0\rangle \langle 0| + e^{i\theta_i} |1\rangle \langle 1|$.

If the environment makes a transition from state $|\mathcal{E}\rangle$ to $|i\rangle$, we let $\mu(\theta_i) = P(|\mathcal{E}\rangle \to |i\rangle)$ the probability of such a transition. Note that $\langle i|\mathcal{E}\rangle = e^{i \arg\langle i|\mathcal{E}\rangle} \sqrt{\mu(\theta_i)}$.

a) What is the initial global state $|\psi_0\rangle$ of the whole system?

b) Show that U is a unitary operator (describe your steps).

c) The state of the system evolves (in discrete time steps say) with a power $n \in \mathbb{N}$ of the operator U as $|\psi_n\rangle = U^n |\psi_0\rangle$. Show that $\mathcal{D}(\theta_i)^n = \mathcal{D}(n\theta_i)$ and deduce that

$$|\psi_n\rangle = \sum_{i=1}^{+\infty} e^{i \arg\langle i|\mathcal{E}\rangle} \sqrt{\mu(\theta_i)}\, |i\rangle \otimes (\mathcal{D}(n\theta_i) |\phi_0\rangle)$$

d) Now let's consider the density matrix of the qubit itself: $\rho_n = \mathrm{tr}_{\mathcal{H}_{\mathcal{E}}} [|\psi_n\rangle \langle \psi_n|]$. First, using only the result of question (a), show that we have initially:

$$\rho_0 = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}$$

e) For any angle $\theta \in \mathbb{R}$, show that we have:

$$\mathcal{D}(\theta)\rho_0\mathcal{D}(\theta)^\dagger = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-i\theta} \\ \alpha^*\beta e^{i\theta} & |\beta|^2 \end{pmatrix}$$

f) Now let's consider $\hat{\theta}$ a random variable taking values $\theta_i in \mathbb{R}$ with probability partial $\mu(\theta_i)$. Use the result of question (c) and (e) to show that the density matrix of the qubit coincide with the following expression:

$$\rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^*\mathbb{E}[e^{-in\hat{\theta}}] \\ \alpha^*\beta\mathbb{E}[e^{in\hat{\theta}}] & |\beta|^2 \end{pmatrix}$$

g) Now say that the values θ_i form a quasicontinuum and that μ is the PDF of a gaussian distribution of mean 0 and variance σ^2. Show that the density matrix of the qubit evolves as:

$$\rho_n = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-\frac{1}{2}\sigma^2 n^2} \\ \alpha^*\beta e^{-\frac{1}{2}\sigma^2 n^2} & |\beta|^2 \end{pmatrix}$$

Calculate $\rho_\infty = \lim_{n\to\infty} \rho_n$.

h) How does the von Neumann entropy of the qubit evolve from initial time $n = 0$ to final time $n \to +\infty$?