Homework 7
Introduction to Quantum Information Processing

Exercise 1 *Partial density matrices: illustrative examples*

- Compute the partial density matrices corresponding to the following pure state:

$$|\Psi\rangle_{AB} = \epsilon|00\rangle + \sqrt{1-\epsilon^2}|11\rangle$$

  with $0 \leq \epsilon \leq 1$. Discuss what happens as ϵ varies and draw the representation in the Bloch ball.

- Now compute the two partial density matrices associated to:

$$|\Phi\rangle_{AB} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

  What is the rank of the partial density matrices ? Why ? Draw the representation on the Bloch ball.

- Now we change one sign in the above:

$$|\Phi'\rangle_{AB} = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$$

  Again compute the two partial reduced density matrices and their rank. Is the rank different or same as above ? Why ?

- Now consider the three qubit state:

$$|W\rangle_{ABC} = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

  Consider the "bipartition" $A \mid BC$; compute the partial density matrices ρ_A and ρ_{BC}. Do the calculations with the Dirac notation. Then write also the final results in tableau (array) representation.

  Same question for the other bipartitions $B \mid AC$ and $C \mid AB$ (hint: no extra calculation is necessary).

Exercise 2 *Tableau computation of partial density matrices*

Sometimes its useful to compute partial traces directly from the tableau (array) representation of the (full) matrix instead of using the Dirac notation (but note that beyond 4×4 matrices this tends to be a bit tedious and Dirac notation is better). Take a 4×4 matrix index as follows $M_{ij;kl}$ with $i, j, k, l \in \{0, 1\}$ (as in class) with lines and columns ordered according to 00, 01, 10, 11. The two partial traces are:

$$M^{(1)}_{jl} = \sum_{m=0,1} M_{mj;ml} \quad \text{and} \quad M^{(1)}_{ik} = \sum_{m=0,1} M_{im;km}$$

What do these formulae amount to do when you are given the tableau (array):

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

Exercise 3 *The difference between a Bell state and a statistical mixture of* $|00\rangle$ *and* $|11\rangle$

We consider a source that distributes to A and B either:

- An *EPR pair* in the perfect Bell state $|B_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$;
- Or distributes a pair of qubits in a *statistical mixture* of states $|00\rangle$, $|11\rangle$ with uniform probabilities 1/2.

This exercise illustrates in many ways that the two kind of situations are completely different.

a) Write down the density matrix ρ_{Bell} associated to the Bell state in Dirac notation as well as in matrix array form (in the computational basis).

b) Write down the density matrix ρ_{stat} associated to the statistical mixture above in Dirac notation as well as in matrix array form (in the computational basis).

c) In a Bell/CSHS experiment one measures the observable

$$\mathcal{B} = A \otimes B + A \otimes B' - A' \otimes B + A' \otimes B'$$

What is the theoretical average if the state when the state is ρ_{Bell} ? (Use results proven in class and no need to reproduce calculations). And now compute the theoretical average if the state is ρ_{stat}. What are the values of the of these two averages for the optimal CSHS-angles $\alpha = 0$, $\alpha' = -\frac{\pi}{4}$, $\beta = \frac{\pi}{8}$, $\beta' = -\frac{\pi}{8}$?

Exercise 4 *Dynamics of 1-qubit density matrix*

In class we showed that the general form of a 1-qubit density matrix is

$$\rho = \frac{1}{2}(I + \vec{a} \cdot \vec{\sigma})$$

where $\vec{a} = a_x, a_y, a_z)$ is a vector in the unit three dimensional ball (the Bloch *ball*) $\|\vec{a}\| \leq 1$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the three usual Pauli matrices.

In a later class we will study the dynamics of qubits driven by magnetic fields. This is an important subject as it at the basis of NMR but also modern quantum information processing technologies. We will see that the time-evolution for a single qubit in a static plus rotating magnetic field in the rotating frame is given by the Hamiltonian

$$H = \frac{\hbar\delta}{2}\sigma_z - \frac{\hbar\omega_1}{2}\sigma_x$$

Here $\omega_1 \propto$ is proportional to the strength of the rotating field and and $\delta = \omega - \omega_0$ is the so-called detuning between the Larmor and rotating magnetic field frequencies.

For the moment you dont have to know where this comes from (as we will see this in later classes). We recall that for a pure state the time evolution is given by:

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle = e^{-\frac{i}{\hbar}tH}|\Psi(0)\rangle$$

From the general definition of a density matrix $\rho = \sum_{i=1}^{K} p_i|\varphi_i\rangle\langle\varphi_i|$ you can infer that the time-evolution is given by (say why?)

$$\rho_t = U_t \rho U_t^\dagger$$

with U_t the evolution operator.

a) Show that the density matrix at time t is of the form

$$\rho_t = \frac{1}{2}(I + \vec{a}(t) \cdot \vec{\sigma})$$

and compute the vector $\vec{a}(t)$.

Hint: We recall the following Euler-type formula seen in previous homeworks:

$$\exp(i\alpha\vec{\sigma} \cdot \vec{n}) = (\cos\alpha)I + i(\sin\alpha)\vec{\sigma} \cdot \vec{n}$$

where α is any angle and $\vec{n}$ is a unit vector.

b) Check that $\|\vec{a}(t)\| = \|\vec{a}\|$. So the vector $\vec{a}(t)$ evolves on a sphere (inside the Bloch ball) of radius given by the initial vector.

c) Find a simple proof of the last statement without ever computing $\vec{a}(t)$.