Homework 11
Introduction to Quantum Information Processing

Exercise 1 *Adapting the Three-Qubit Repetition Code to Correct Phase-Flip Errors*

The standard three-qubit repetition code encodes a single qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \longmapsto \quad \alpha|000\rangle + \beta|111\rangle,$$

and protects against a single bit-flip (Pauli-X) error by measuring pairwise parities of the three physical qubits. However, this code does not protect against phase-flip (Z) errors, which do not change the computational-basis parities that the code checks. One approach to obtain phase-flip protection is to work in the Hadamard basis.

Consider the encoding

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle \quad \longmapsto \quad \alpha|+++\rangle + \beta|---\rangle$$

In the standard (bit-flip) repetition code one measures the observable

$$Z_1Z_2, \qquad Z_2Z_3.$$

a) Determine the observable that should be measured to detect phase-flip errors in the Hadamard basis repetition code.

b) For each possible single-qubit Z error, give the expected syndrome outcomes.

c) Show that this encoding protects against a single Z error on any one of the three qubits using majority decoding.

d) Adapt the circuit described in class to detect and correct the errors.

e) Determine the logical operator X_L and Z_L operators that verifies

$$X_L\,|+++\rangle = |---\rangle\,, \quad X_L\,|---\rangle = |+++\rangle$$
$$Z_L\,|+++\rangle = |+++\rangle\,, \quad Z_L\,|---\rangle = -\,|---\rangle$$

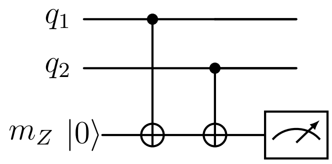
Exercise 2 *Syndrome measurement for 2 qubits*

Consider a 2-qubit system with data qubits q_1 and q_2, and two measurement (ancilla) qubits m_Z and m_X. We want to measure the operators (stabilizers) on the data qubits:

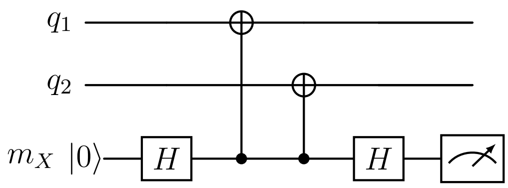
$$S_Z = Z_1 \otimes Z_2, \qquad S_X = X_1 \otimes X_2$$

The measurement scheme uses CNOTs as follows:

- To measure S_Z: initialize m_Z in $|0\rangle$, apply CNOTs controlled on q_1 and q_2 to m_Z, then measure m_Z in the computational basis.



- To measure S_X: initialize m_X in $|+\rangle$ thanks to an Hadamard gate, apply CNOTs controlled on m_X to q_1 and q_2, then measure m_X in the X-basis by applying another Hadamard gate.



a) Prove that these circuits indeed implement a measurement of the operators S_Z and S_X.

b) If q_1 and q_2 are in a state of the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, why measuring S_Z can tell us if they are identical or not? Deduce that S_Z can detect a bit-flip error.

c) Explain why measuring S_X can also tell us if q_1 and q_2 are identical or not when they are expressed in the Hadamard basis $\{|+\rangle, |-\rangle\}$. Deduce that S_X can detect a phase-flip error.

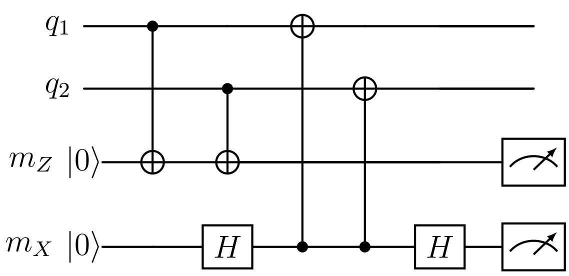
d) Write the 4 Bell states

$$|\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \quad |\Psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$$

in the Hadamard basis $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$.

e) Show that the four Bell states are eigenvectors of both operators S_Z and S_X, and identify the corresponding eigenvalues.

Combining the two measurement circuits, we obtain the following full circuit for measuring S_Z and S_X in one cycle:



f) Starting from a general data state

$$|\psi\rangle_{q_1,q_2} = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle,$$

compute the combined state of the data and measurement qubits after one cycle (before measurement).

g) Compute the possible measurement outcomes (syndrome) if q_1, q_2 are in each of the four Bell states. If the initial state is $|\Phi^+\rangle_{q_1,q_2}$, identify the errors which correspond to each combination of m_Z and m_X measurement results.

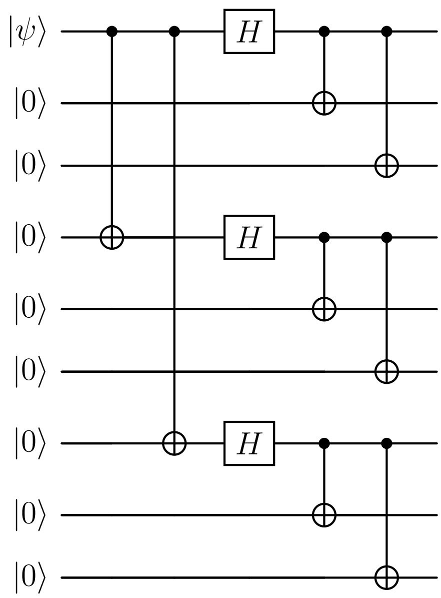
This scheme can be used for error detection and correction if the 2 data-qubits are prepared in the state $|\Phi^+\rangle$. The system is however too small to implement a complete logical qubit.

Exercise 3 *Shor 9 qubit code implementation*

The Shor 9-qubit code encodes one logical qubit into 9 physical qubits as follows:

$$|0\rangle_L = \frac{(|000\rangle + |111\rangle)^{\otimes 3}}{2\sqrt{2}}, \quad |1\rangle_L = \frac{(|000\rangle - |111\rangle)^{\otimes 3}}{2\sqrt{2}}.$$

a) Let $|\psi\rangle$ be the state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Verify that the following circuit can encode an arbitrary qubit state $|\psi\rangle_L = \alpha\,|0\rangle_L + \beta\,|1\rangle_L$ with the Shor code.



If the logical gates are implemented by applying the same physical gate to each of the 9 physical qubits, we call this a transversal implementation. Transversal gates are important in fault-tolerant quantum computing because they are easy to implement and help prevent the spread of errors.

b) Suggest transversal logical operators $\overline{X}$ and $\overline{Z}$ that act on the logical qubit such that

$$\overline{X}\,|0\rangle_L = |1\rangle_L\,, \quad \overline{X}\,|1\rangle_L = |0\rangle_L\,, \quad \overline{Z}\,|0\rangle_L = |0\rangle_L\,, \quad \overline{Z}\,|1\rangle_L = -\,|1\rangle_L\,.$$

Deduce logical operators for $\overline{Y}$ and $\overline{H}$.