Homework 10

Introduction to Quantum Information Processing

Exercise 1 *Errors in the Bloch ball*

The quantum state of a qubit can be represented as a point in the Bloch ball. Let $\rho = \frac{1}{2}(\mathbf{I} + \vec{r} \cdot \vec{\sigma})$ be the density matrix of a qubit where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of the Pauli matrices. The corresponding point in the Bloch ball is given by the vector $\vec{r} = (r_x, r_y, r_z)$.

a) Suppose that the qubit undergoes a bit-flip error with probability p. What is the new vector $\vec{r'}$ in the Bloch ball? Which axis is unaffected?

b) Same question for a phase-flip error with probability q.

We define the depolarising channel as the quantum channel that with probability $p/3$ applies a bit-flip (X error), a phase-flip (Z error) or a bit-and-phase flip (Y error) to the qubit, and leaves it unchanged with probability $1 - p$. The action of the depolarising channel on a density matrix ρ is given by:

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z).$$

c) Consider the depolarising channel when $p = \frac{3}{4}$, i.e., the channel $\mathcal{E}(\rho) = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$. How is the Bloch vector affected by this channel?

d) If the qubit is subject to a depolarizing noise with probability p, show that the new vector is given by $\vec{r'} = (1 - \frac{4p}{3})\vec{r}$.

e) Suppose a qubit is first subject to a bit-flip with probability p_X, then a phase-flip with probability p_Z. For which values of p_X, p_Z is this equivalent to a depolarising channel ?

f) For an amplitude damping process with $|0\rangle \to |0\rangle$ and $|1\rangle \to \sqrt{1-\gamma}\,|1\rangle + \sqrt{\gamma}\,|0\rangle$, describe qualitatively the transformation of the Bloch vector.

Exercise 2 *Three-qubit repetition code*

Consider the three-qubit repetition code with logical states $|0\rangle_L = |000\rangle$ and $|1\rangle_L = |111\rangle$.

a) How are the logical states sensible to phase-flip errors? Is it the same for a logical qubit $|\phi\rangle = \alpha\,|0\rangle_L + \beta\,|1\rangle_L$?

b) Same question for Y errors.

c) Suppose that each physical qubit in the 3-qubit code is independently subject to a depolarizing channel with probability p. For a logical state, can you simplify the analysis by considering only bit-flip errors? Justify your answer. What is then the effective bit-flip probability p_X per qubit?

d) For a logical state, derive the probability that:

- the logical state is unaffected, (error can occurs but are corrected)
- a logical bit-flip error occurs, i.e. the majority decoding fails.

e) Plot the logical error rate p_L (probability of failure) as a function of p. Determine the regime in which encoding provides a net improvement, i.e. $p_L < p$.