Additional exercises to test yourself
Introduction to Quantum Information Processing

Exercise 1 *Entropy of reduced states for mixed global states*

If $|\psi\rangle_{AB}$ is pure, then

$$S(\rho_A) = S(\rho_B).$$

a) Is this still true if, instead of $|\psi\rangle_{AB}$, the global state is a mixed (i.e., not pure) state?

b) Find an example of ρ_{AB} (mixed state) such that

$$S(\rho_A) \neq S(\rho_B).$$

Hint: You may use the technique shown in last exercise class for computing ρ_A, ρ_B from ρ_{AB} for a 2–qubit ρ_{AB}.

Exercise 2 *Basis changes*

Let

$$|\psi\rangle = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

be a 2–qubit state in the computational basis.

a) What are all the Hadamard basis vectors on 1 qubit?

b) How do you write this state in the Hadamard basis?

Hint: Use the expression

$$|\psi\rangle = \sum_{i=1}^{m} \langle\phi_i|\psi\rangle \, |\phi_i\rangle$$

for any orthonormal set $\{|\phi_i\rangle\}_{i=1}^{m}$.

c) (Bonus) Prove the above expression.

Exercise 3 *Criterion for pure states*

a) Prove that a density matrix ρ corresponds to a pure state if and only if $\text{Tr}(\rho^2) = 1$.

b) Consider a density matrix on a two-qubit system ρ_{AB} and its reduced states $\rho_A = \mathrm{Tr}_B(\rho_{AB})$ and $\rho_B = \mathrm{Tr}_A(\rho_{AB})$. Under what conditions is it guaranteed that $\mathrm{Tr}(\rho_A^2) = \mathrm{Tr}(\rho_B^2)$? Find an example where this is not the case.

Exercise 4 *Measurement*

a) Show that the average value of the observable $X \otimes Z$ for a two–qubit system measured in the state

$$|\psi\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$$

is zero.

b) Write the matrix form of the state

$$\rho = p\, |\psi\rangle\langle\psi| + (1-p)\, |0\rangle\langle 0| \otimes |+\rangle\langle +|$$

where $0 \le p \le 1$. Compute the expectation value of the observable $X \otimes Z$ measured on ρ and determine for which values of p the expectation value is zero.

Exercise 5 *Hermitian Operators and Tensor Products*

Let A and B be Hermitian operators acting on 2-dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

a) Show that the tensor product $A \otimes B$ is Hermitian, i.e., prove that

$$(A \otimes B)^\dagger = A \otimes B.$$

b) If $|\psi\rangle \in \mathcal{H}_1$ is an eigenvector of A with eigenvalue λ, and $|\phi\rangle \in \mathcal{H}_2$ is an eigenvector of B with eigenvalue μ, show that

$$|\psi\rangle \otimes |\phi\rangle$$

is an eigenvector of $A \otimes B$ with eigenvalue $\lambda\mu$. How many eigenvalues does $A \otimes B$ have in total?

c) Let

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Verify explicitly that A and B are Hermitian. Compute $A \otimes B$ and check that it is Hermitian.

d) Find the eigenvalues of $A \otimes B$ in terms of the eigenvalues of A and B.

Exercise 6 *Partial Measurement*

Consider the three-qubit state

$$|\Psi\rangle = \frac{1}{2}\Big(|001\rangle + e^{i\pi/4}|011\rangle - |100\rangle + |010\rangle\Big).$$

You measure only the qubit 2 (the middle qubit) in the computational basis $\{|0\rangle, |1\rangle\}$.

a) Compute the probabilities to measure each outcome $0, 1$ and verify that $p(0) + p(1) = 1$. To do this, you can define the projectors acting on the three-qubit space as

$$P_0 = I \otimes |0\rangle\langle 0| \otimes I, \quad P_1 = I \otimes |1\rangle\langle 1| \otimes I.$$

and compute the probabilities as the expectation values of these projectors.

b) For each outcome $k = 0, 1$, compute the state after measurement. Write each resulting state explicitly as a superposition of three-qubit basis states.

c) For each resulting state, determine whether qubits 1 and 3 are entangled or separable