# CIVIL-457
# Exercise

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1. The two-region MFD system dynamics are given as follows:

$$\dot{n}_{11}(t) = -\frac{n_{11}(t)}{n_1(t)} G_1(n_1(t)) + \frac{n_{21}(t)}{n_2(t)} G_2(n_2(t)) u_{21}(t) + q_{11}(t)$$
$$\dot{n}_{12}(t) = -\frac{n_{12}(t)}{n_1(t)} G_1(n_1(t)) u_{12}(t) + q_{12}(t)$$
$$\dot{n}_{21}(t) = -\frac{n_{21}(t)}{n_2(t)} G_2(n_2(t)) u_{21}(t) + q_{21}(t)$$
$$\dot{n}_{22}(t) = -\frac{n_{22}(t)}{n_2(t)} G_2(n_2(t)) + \frac{n_{12}(t)}{n_1(t)} G_1(n_1(t)) u_{12}(t) + q_{22}(t)$$

Given the steady-state regional accumulations $n_1(t) = \bar{n}_1,\ n_2(t) = \bar{n}_2$, and steady-state inflows $q_{11}(t) = \bar{q}_{11},\ q_{12}(t) = \bar{q}_{12},\ q_{21}(t) = \bar{q}_{21},\ q_{22}(t) = \bar{q}_{22}$.

(a) Please write down the steady-state equations. *Hint: The steady-state condition satisfies* $\dot{\bar{n}}_{ij} = 0,\ \forall i, j \in \{1, 2\}$.

(b) Suppose that the following equalities hold

$$\bar{n}_1 = \bar{n}_{11} + \bar{n}_{12} \tag{1}$$
$$\bar{n}_2 = \bar{n}_{21} + \bar{n}_{22} \tag{2}$$

In conjunction with the steady-state equations, please derive the steady-state accumulations $\bar{n}_{11},\ \bar{n}_{12},\ \bar{n}_{21},\ \bar{n}_{22}$.

(c) Please derive the steady-state control inputs $\bar{u}_{12},\ \bar{u}_{21}$.

(d) An open question: please give a thought of the relationship between the steady-state supply (capacity, $G_i(\bar{n}_i)$) and demand (inflow, $\bar{q}_{ii} + \bar{q}_{ji}$). Can $G_i(\bar{n}_i) < \bar{q}_{ii} + \bar{q}_{ji}$? What does this imply for designing a control strategy?

**Ans:**

(a) The steady-state equations are given as

$$0 = \dot{\bar{n}}_{11} = -\frac{\bar{n}_{11}}{\bar{n}_1} G_1(\bar{n}_1) + \frac{\bar{n}_{21}}{\bar{n}_2} G_2(\bar{n}_2)\bar{u}_{21} + \bar{q}_{11} \tag{3}$$

$$0 = \dot{\bar{n}}_{12} = -\frac{\bar{n}_{12}}{\bar{n}_1} G_1(\bar{n}_1)\bar{u}_{12} + \bar{q}_{12} \tag{4}$$

$$0 = \dot{\bar{n}}_{21} = -\frac{\bar{n}_{21}}{\bar{n}_2} G_2(\bar{n}_2)\bar{u}_{21} + \bar{q}_{21} \tag{5}$$

$$0 = \dot{\bar{n}}_{22} = -\frac{\bar{n}_{22}}{\bar{n}_2} G_2(\bar{n}_2) + \frac{\bar{n}_{12}}{\bar{n}_1} G_1(\bar{n}_1)\bar{u}_{12} + \bar{q}_{22} \tag{6}$$

(b) We sum both sides of (3) and (5) and obtain

$$0 = -\frac{\bar{n}_{11}}{\bar{n}_1} G_1(\bar{n}_1) + \bar{q}_{11} + \bar{q}_{21} \tag{7}$$

Similarly, summing both sides of (4) and (6) yields

$$0 = -\frac{\bar{n}_{22}}{\bar{n}_2} G_2(\bar{n}_2) + \bar{q}_{12} + \bar{q}_{22} \tag{8}$$

From (7) and (8), we have

$$\bar{n}_{11} = \frac{\bar{n}_1(\bar{q}_{11} + \bar{q}_{21})}{G_1(\bar{n}_1)} \tag{9}$$

$$\bar{n}_{22} = \frac{\bar{n}_2(\bar{q}_{12} + \bar{q}_{22})}{G_2(\bar{n}_2)} \tag{10}$$

In conjunction with (1) and (2), we have

$$\bar{n}_{12} = \bar{n}_1 - \frac{\bar{n}_1(\bar{q}_{11} + \bar{q}_{21})}{G_1(\bar{n}_1)} = \bar{n}_1 \frac{G_1(\bar{n}_1) - (\bar{q}_{11} + \bar{q}_{21})}{G_1(\bar{n}_1)} \tag{11}$$

$$\bar{n}_{22} = \bar{n}_2 - \frac{\bar{n}_2(\bar{q}_{12} + \bar{q}_{22})}{G_2(\bar{n}_2)} = \bar{n}_2 \frac{G_2(\bar{n}_2) - (\bar{q}_{12} + \bar{q}_{22})}{G_2(\bar{n}_2)} \tag{12}$$

(c) We then substitute (11) and (12) into (4) and (5) respectively,

$$0 = -\bar{n}_1 \frac{G_1(\bar{n}_1) - (\bar{q}_{11} + \bar{q}_{21})}{G_1(\bar{n}_1)} \cdot \frac{G_1(\bar{n}_1)}{\bar{n}_1} \cdot \bar{u}_{12} + \bar{q}_{12}$$

$$\Rightarrow \bar{u}_{12} = \frac{\bar{q}_{12}}{G_1(\bar{n}_1) - (\bar{q}_{11} + \bar{q}_{21})} \tag{13}$$

$$0 = -\bar{n}_2 \frac{G_2(\bar{n}_2) - (\bar{q}_{12} + \bar{q}_{22})}{G_2(\bar{n}_2)} \cdot \frac{G_2(\bar{n}_2)}{\bar{n}_2} \cdot \bar{u}_{21} + \bar{q}_{21}$$

$$\Rightarrow \bar{u}_{21} = \frac{\bar{q}_{21}}{G_2(\bar{n}_2) - (\bar{q}_{12} + \bar{q}_{22})} \tag{14}$$