CIVIL-457
Exercise

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1. The two-region MFD system dynamics are given as follows:

$$\begin{aligned}
\dot{n}_{11}(t) &= -\frac{n_{11}(t)}{n_1(t)}G_1(n_1(t)) + \frac{n_{21}(t)}{n_2(t)}G_2(n_2(t))u_{21}(t) + q_{11}(t) \\
\dot{n}_{12}(t) &= -\frac{n_{12}(t)}{n_1(t)}G_1(n_1(t))u_{12}(t) + q_{12}(t) \\
\dot{n}_{21}(t) &= -\frac{n_{21}(t)}{n_2(t)}G_2(n_2(t))u_{21}(t) + q_{21}(t) \\
\dot{n}_{22}(t) &= -\frac{n_{22}(t)}{n_2(t)}G_2(n_2(t)) + \frac{n_{12}(t)}{n_1(t)}G_1(n_1(t))u_{12}(t) + q_{22}(t)
\end{aligned}$$

Given the steady-state regional accumulations $n_1(t) = \bar{n}_1,\ n_2(t) = \bar{n}_2$, and steady-state inflows $q_{11}(t) = \bar{q}_{11},\ q_{12}(t) = \bar{q}_{12},\ q_{21}(t) = \bar{q}_{21},\ q_{22}(t) = \bar{q}_{22}$.

(a) Please write down the steady-state equations. *Hint: The steady-state condition satisfies* $\dot{\bar{n}}_{ij} = 0,\ \forall i, j \in \{1, 2\}$.

(b) Suppose that the following equalities hold

$$\bar{n}_1 = \bar{n}_{11} + \bar{n}_{12} \tag{1}$$

$$\bar{n}_2 = \bar{n}_{21} + \bar{n}_{22} \tag{2}$$

In conjunction with the steady-state equations, please derive the steady-state accumulations $\bar{n}_{11},\ \bar{n}_{12},\ \bar{n}_{21},\ \bar{n}_{22}$.

(c) Please derive the steady-state control inputs $\bar{u}_{12},\ \bar{u}_{21}$.

(d) An open question: please give a thought of the relationship between the steady-state supply (capacity, $G_i(\bar{n}_i)$) and demand (inflow, $\bar{q}_{ii} + \bar{q}_{ji}$). Can $G_i(\bar{n}_i) < \bar{q}_{ii} + \bar{q}_{ji}$? What does this imply for designing a control strategy?