CIVIL 449: Nonlinear Analysis of Structures

School of Architecture, Civil & Environmental Engineering
Civil Engineering Institute

Truss, frame and zero length elements

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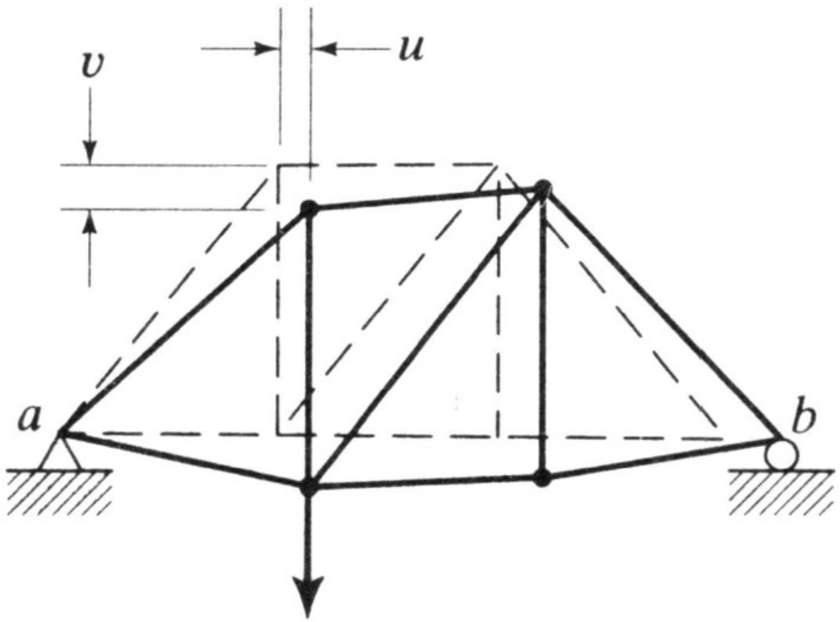
Objectives of today's lecture

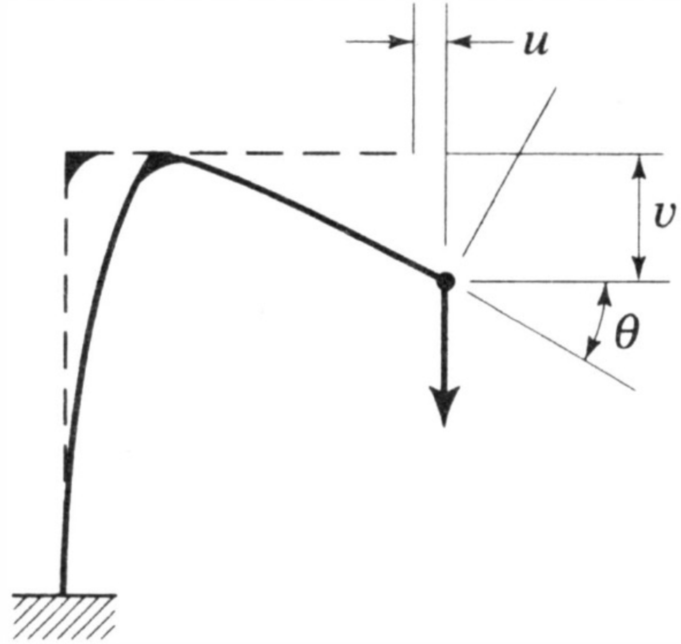
- To introduce:
  - Terminology & revision
  - Truss elements
  - Frame (beam-column) elements
  - Zero-length elements
  - Element transformations (from local to global coordinate system)

Degrees of freedom
-Displacement field

Pin-jointed plane truss

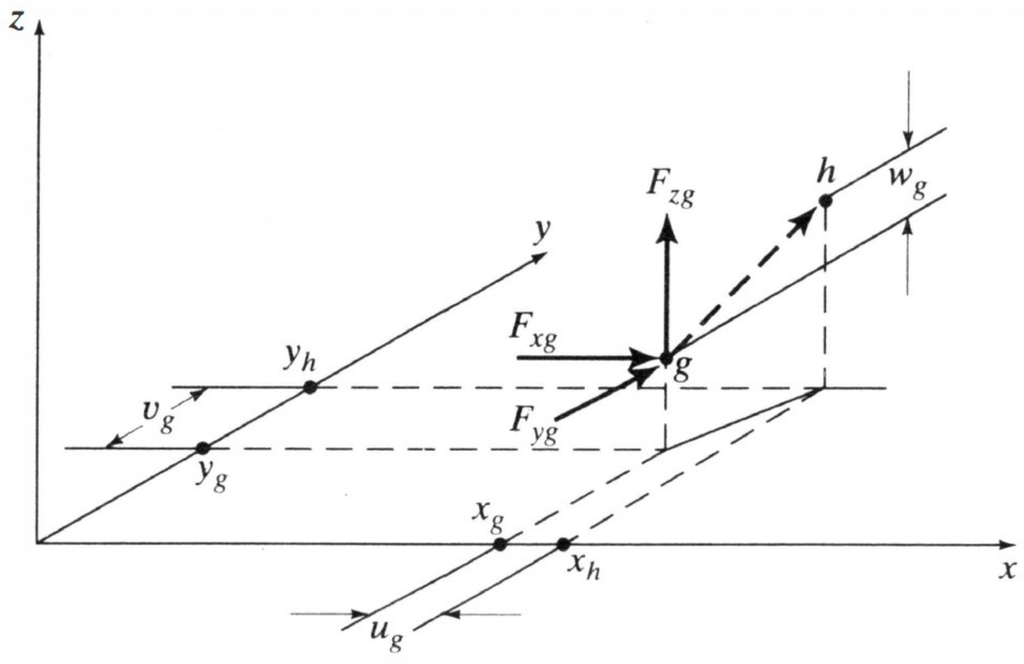


Plane frame

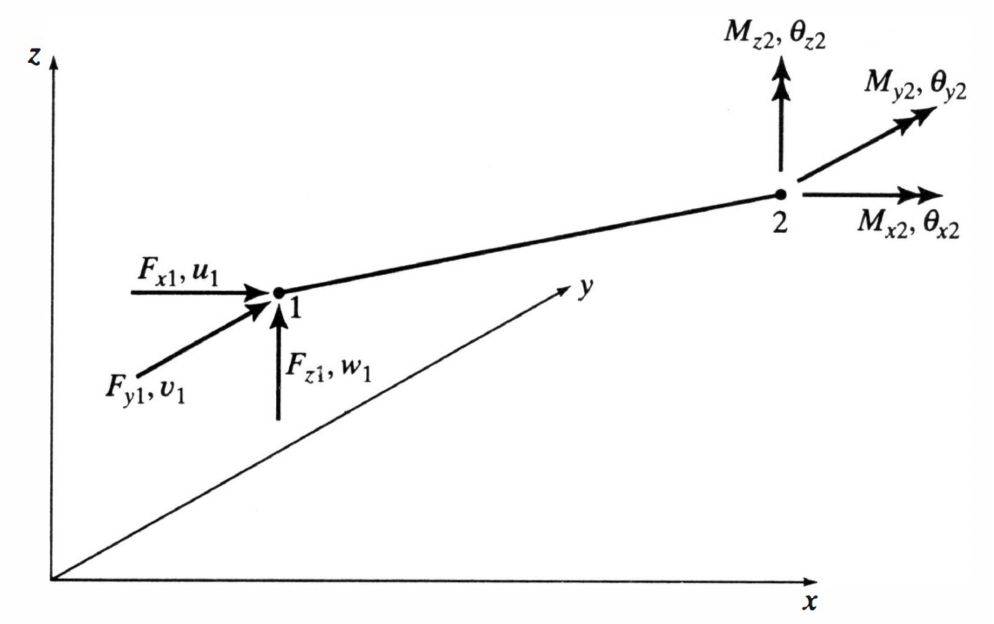


Coordinate systems and conditions of analysis

Displacement from point *g* to point *h*

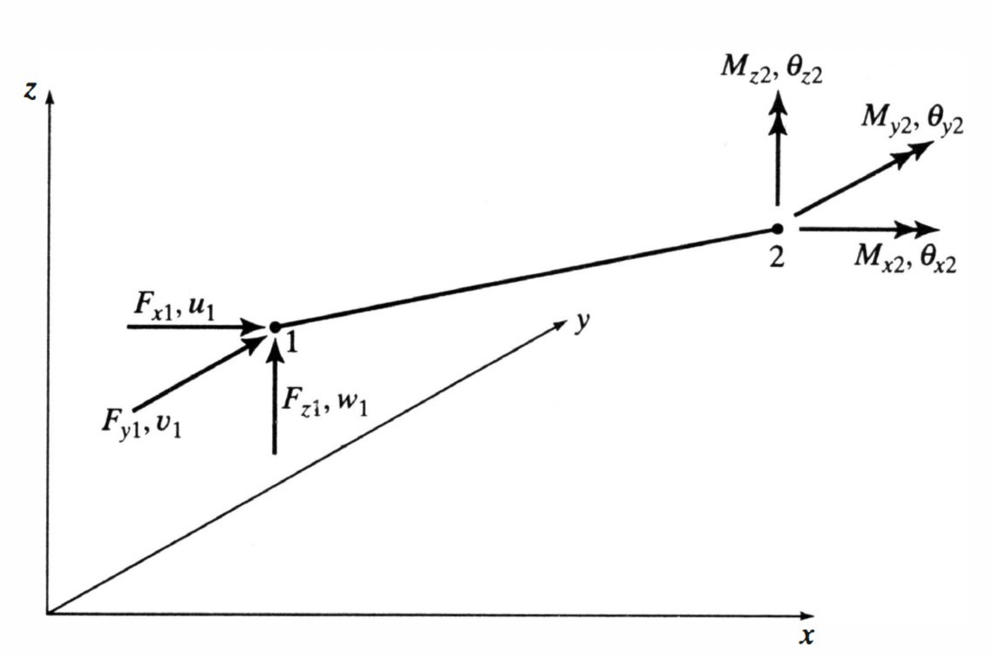


Forces, moments and corresponding displacements



$$\theta_{x2} = \left.\frac{\vartheta w}{\vartheta y}\right|_2 \qquad \theta_{y2} = \left.\frac{\vartheta w}{\vartheta x}\right|_2 \qquad \theta_{z2} = \left.\frac{\vartheta v}{\vartheta x}\right|_2$$

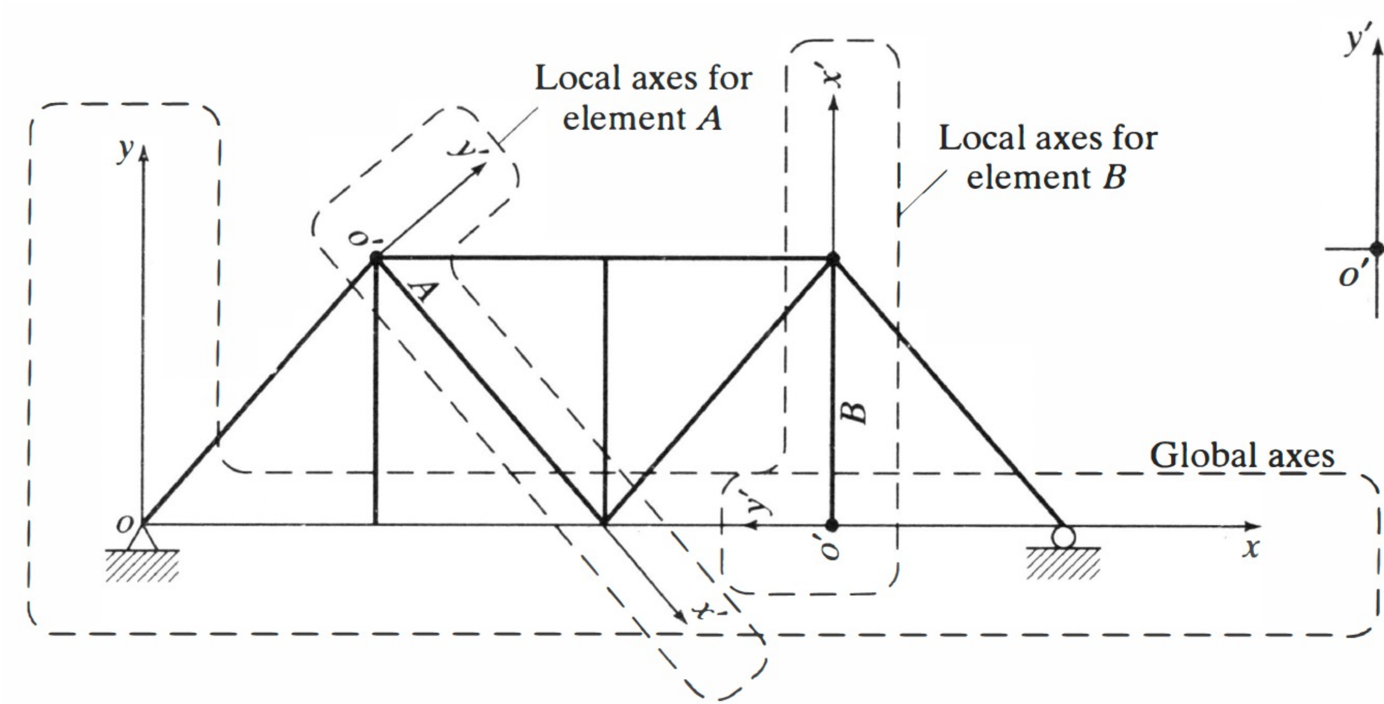
Coordinate systems and conditions of analysis (2)

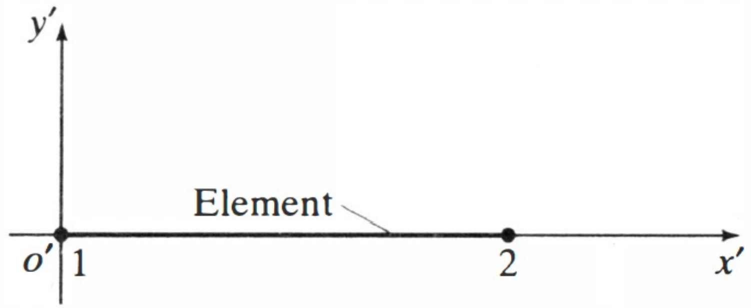


$$\{\mathbf{F}\} = \lfloor F_{x1} \quad F_{y1} \quad F_{z1} \quad M_{x2} \quad M_{y2} \quad M_{z2} \rfloor^{\mathrm{T}}$$

$$\{\boldsymbol{\Delta}\} = \lfloor u_1 \quad v_1 \quad w_1 \quad \theta_{x2} \quad \theta_{y2} \quad \theta_{z2} \rfloor^{\mathrm{T}}$$

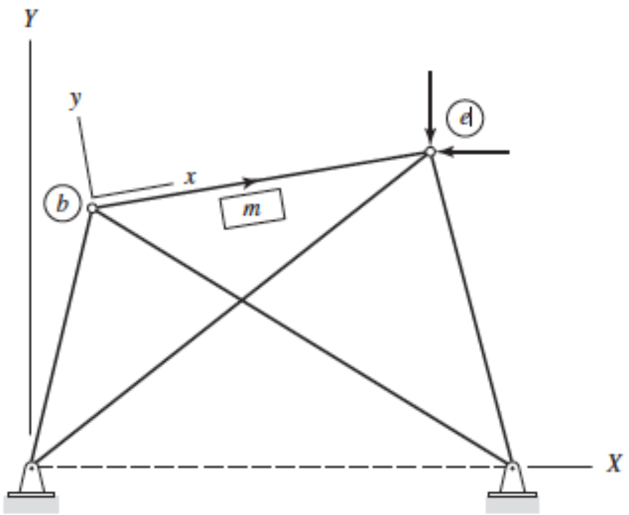
Coordinate systems and conditions of analysis (3)

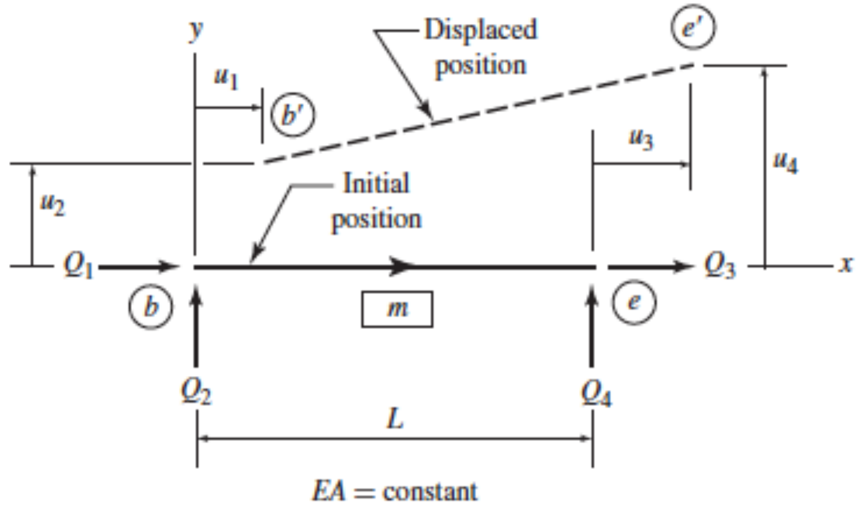




The truss element

We will deal with prismatic members (constant axial rigidity, EA)

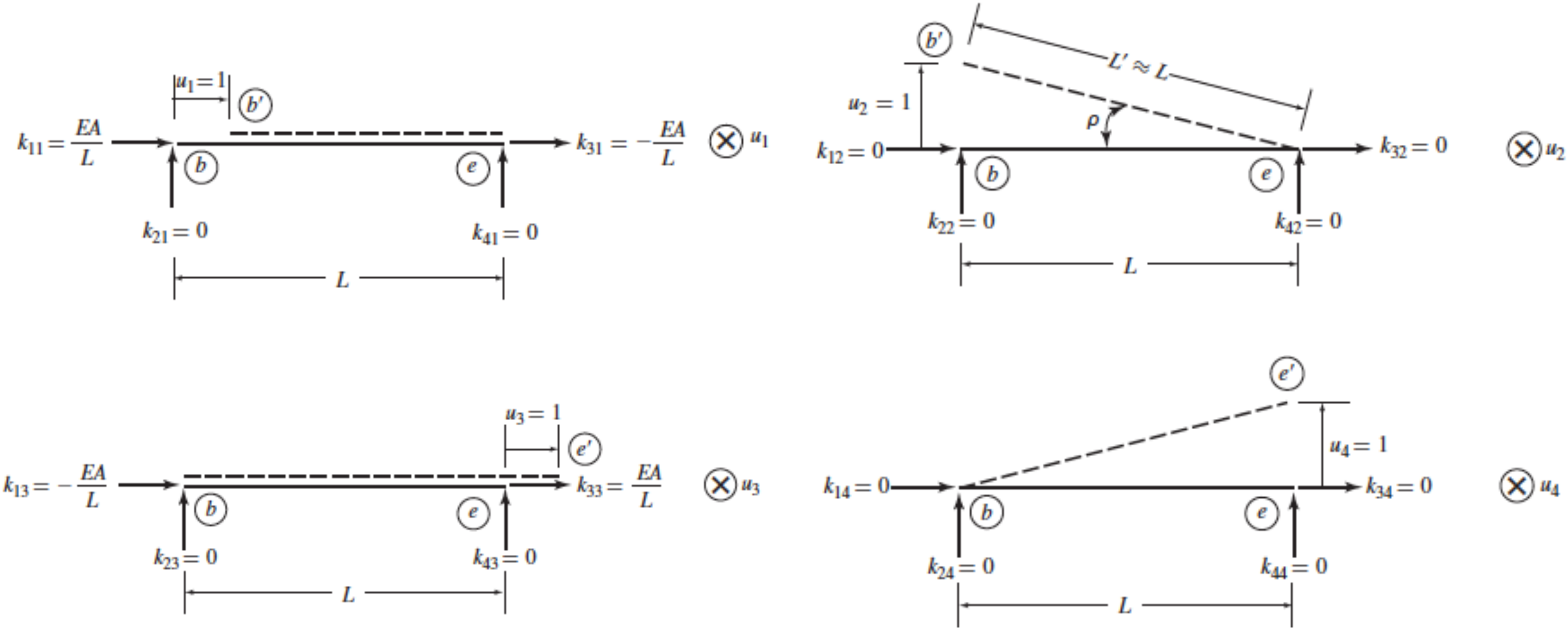




Member end displacements, u_1, u_2, u_3, u_4

Member end forces, Q_1, Q_2, Q_3, Q_4

The truss element



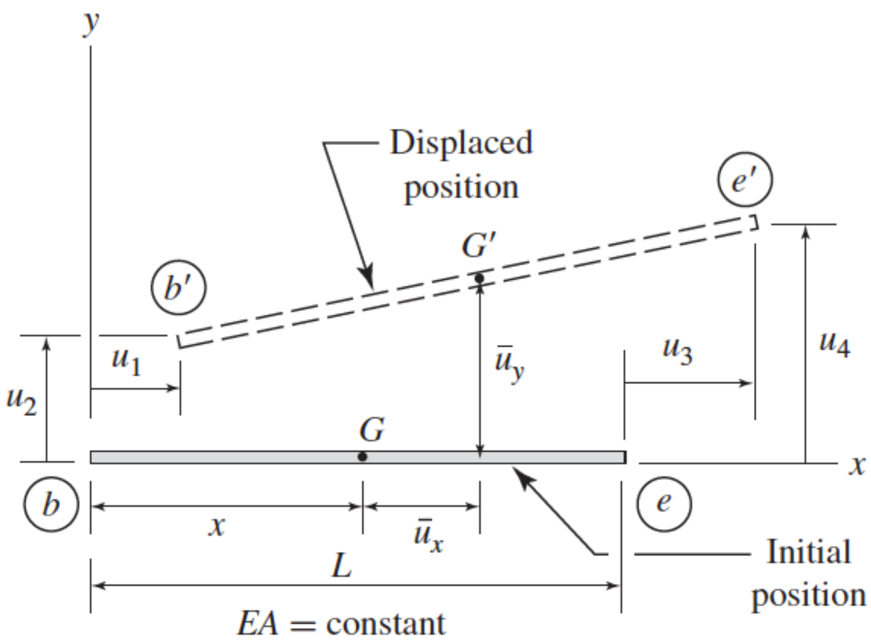
Stiffness matrix of a plane truss element

$$\{\mathbf{F}\} = [\mathbf{k}]\{\boldsymbol{\Delta}\}$$

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{bmatrix} \frac{EA}{L} & 0 & -\frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{EA}{L} & 0 & \frac{EA}{L} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Finite element formulation for deriving the stiffness matrix

In the finite element method, member stiffness relations are based on assumed variations of displacements within members, such displacement variations are referred to as the displacement interpolation functions, N_i.



$$\mathbf{\Delta} = \sum_{i=1}^{N} N_i \Delta_i = \lfloor \mathbf{N} \rfloor \{\mathbf{\Delta}\}$$

Use of shape functions

- Shape functions are usually complete polynomials in which n is the degree of freedom of the polynomial.

- The polynomial used for a particular displacement function should be of such a degree that all of its coefficients can be evaluated by applying the available boundary conditions; that is, $n = n_{bc} - 1$

$$\bar{u}_x = a_0 + a_1 x$$

At $x = 0, \bar{u}_x = u_1$ $\qquad a_0 = u_1$

At $x = L, \bar{u}_x = u_3$ $\qquad u_3 = u_1 + a_1 L \rightarrow \qquad a_1 = \dfrac{u_3 - u_1}{L}$

Use of shape functions (2)

The following expression is obtained for a truss element in x direction,

$$\bar{u}_x = u_1 + \frac{u_3 - u_1}{L} x$$

or

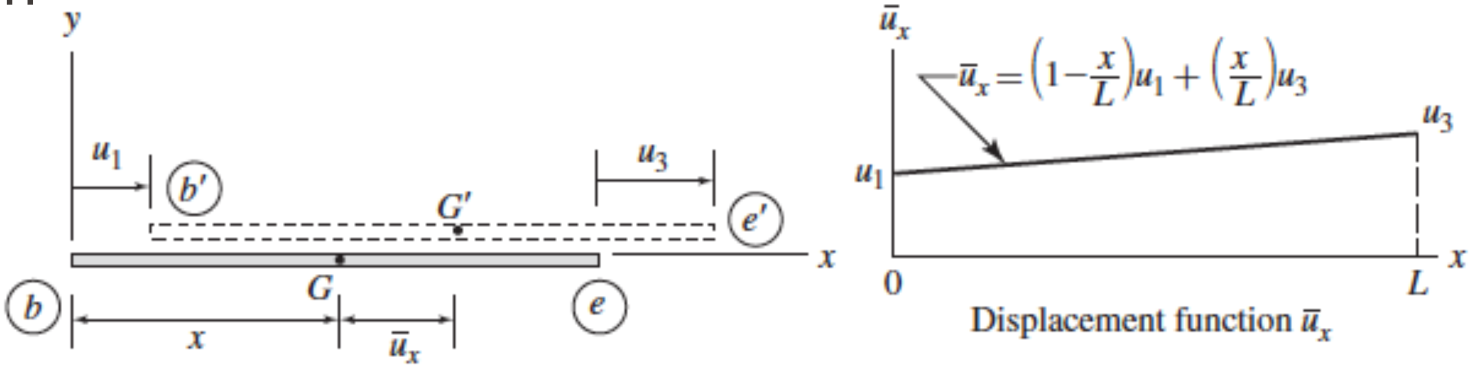
$$\bar{u}_x = \left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_3$$

Similarly, in the y direction,

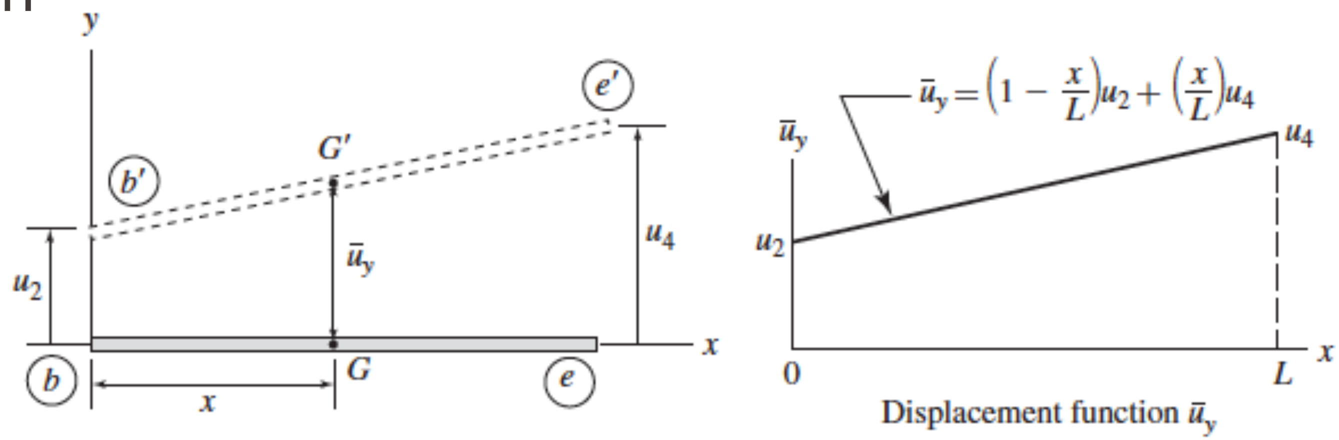
$$\bar{u}_y = \left(1 - \frac{x}{L}\right) u_2 + \left(\frac{x}{L}\right) u_4$$

Finite element formulation

x-direction



y-direction



Shape functions

The displacement functions can be expressed as follows:

$$\bar{u}_x = N_1 u_1 + N_3 u_3$$

$$\bar{u}_y = N_2 u_2 + N_4 u_4$$

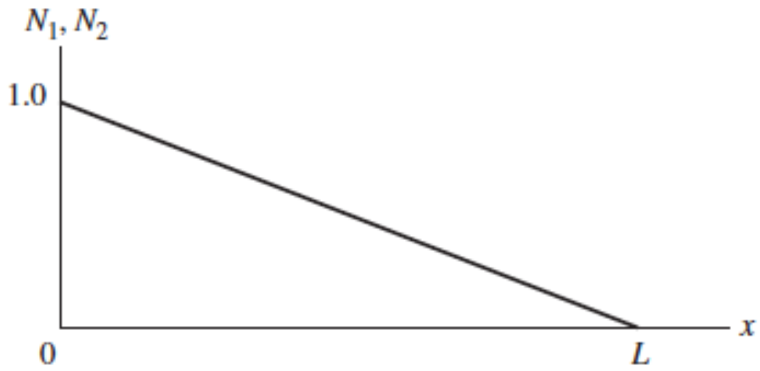
Shape functions

$$N_1 = N_2 = 1 - \frac{x}{L} \qquad N_3 = N_4 = \frac{x}{L}$$

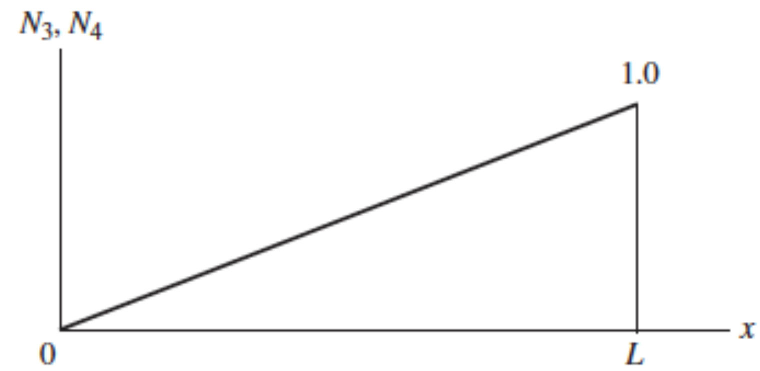
The displacement field in matrix form,

$$\begin{Bmatrix} \bar{u}_x \\ \bar{u}_y \end{Bmatrix} = \underbrace{\begin{bmatrix} N_1 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_4 \end{bmatrix}}_{\mathbf{N}} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Properties of shape functions



(a) Shape Functions N_1 ($u_1 = 1, u_2 = u_3 = u_4 = 0$)
and N_2 ($u_2 = 1, u_1 = u_3 = u_4 = 0$)



(b) Shape Functions N_3 ($u_3 = 1, u_1 = u_2 = u_4 = 0$)
and N_4 ($u_4 = 1, u_1 = u_2 = u_3 = 0$)

$$\sum_{i=1}^{N} N_i = 1$$

Computing strains from end displacements

Having established expressions for the displacement of any point on an element terms of the element node-point displacements we can employ these expressions in the principle of virtual displacements to produce element stiffness equations.

Plane trusses

$$e = \frac{d\bar{u}_x}{dx} = \underbrace{\begin{bmatrix} \frac{d}{dx} & 0 \end{bmatrix}}_{\mathbf{D}} \begin{Bmatrix} \bar{u}_x \\ \bar{u}_y \end{Bmatrix}$$

$\mathbf{D}$ Differential operator matrix

Computing strains from end displacements (2)

$$e = \mathbf{D}\begin{Bmatrix} \bar{u}_x \\ \bar{u}_y \end{Bmatrix} = \mathbf{D}\begin{bmatrix} N_1 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_4 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \mathbf{DN}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = (\mathbf{DN})\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

The matrix $(\mathbf{DN})$ is called the member strain-displacement matrix

$$\mathbf{B} = \mathbf{DN} = \begin{bmatrix} \frac{d}{dx} & 0 \end{bmatrix}\begin{bmatrix} N_1 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_4 \end{bmatrix} = \begin{bmatrix} \frac{dN_1}{dx} & 0 & \frac{dN_3}{dx} & 0 \end{bmatrix}$$

Computing strains from end displacements (3)

Recall,

$$N_1 = 1 - \frac{x}{L} \qquad N_3 = \frac{x}{L}$$

Therefore,

$$\mathbf{B} = \begin{bmatrix} \frac{dN_1}{dx} & 0 & \frac{dN_3}{dx} & 0 \end{bmatrix} = \begin{bmatrix} \frac{d}{dx}\left(1 - \frac{x}{L}\right) & 0 & \frac{d}{dx}\left(\frac{x}{L}\right) & 0 \end{bmatrix}$$

Finally,

$$\mathbf{B} = \frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}$$

Recapping…

The strain along the element length is expressed in terms of end displacements,

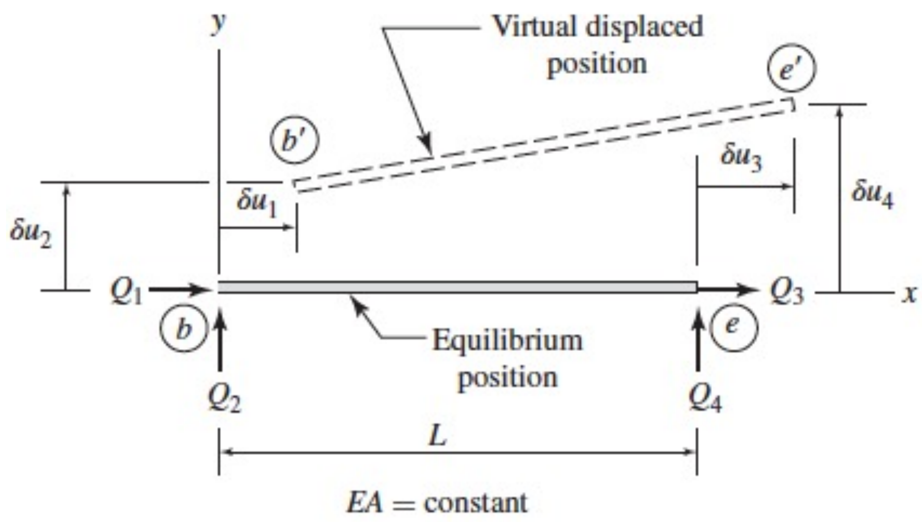
$$e = \mathbf{B}\{\boldsymbol{\Delta}\}$$

The stress,

$$\sigma = Ee = E\mathbf{B}\{\boldsymbol{\Delta}\}$$

Member stiffness matrix
-Plane truss

We can establish the relationship between member end forces $\mathbf{F}$ and end displacements, by applying the principle of virtual work for deformable bodies



$$\delta W_e = Q_1 \delta u_1 + Q_2 \delta u_2 + Q_3 \delta u_3 + Q_4 \delta u_4$$

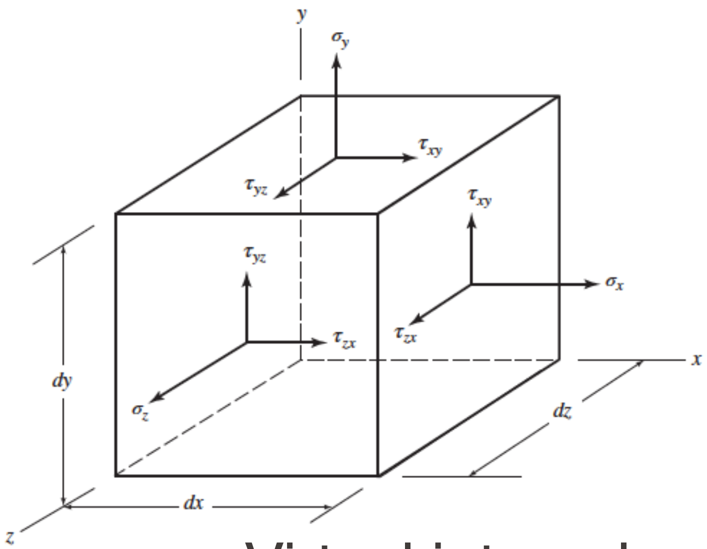
$$\delta W_e = [\delta u_1 \quad \delta u_2 \quad \delta u_3 \quad \delta u_4] \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

$$\delta W_e = \delta \boldsymbol{\Delta}^T \mathbf{F}$$

Member stiffness matrix (2)

-Plane truss

From virtual work of deformable bodies,



From virtual work of deformable bodies,

$$\delta W_e = \delta W_i$$

Real force = stress x area $= \sigma_x(dydz)$

Virtual displacement= strain x length $= (\delta\varepsilon_x)dx$

Virtual internal work = real force x virtual displacement

$$= \sigma_x(dydz)(\delta\varepsilon_x)dx \; = (\delta\varepsilon_x\sigma_x)dV$$

$$\delta W_e = \int_V \delta\varepsilon^T \sigma dV$$

Member stiffness matrix (3)
-Plane truss

Therefore, for the plane truss

$$\delta W_e = \delta \mathbf{\Delta}^T \mathbf{F} = \int_V \delta \varepsilon^T \sigma dV$$

$$\delta W_e = \delta \mathbf{\Delta}^T \mathbf{F} = \int_V (\mathbf{B}\delta\mathbf{\Delta})^T E \mathbf{B} dV \mathbf{\Delta} \qquad (\mathbf{B}\delta\mathbf{\Delta})^T = (\delta\mathbf{\Delta})^T \mathbf{B}^T$$

$$\delta \mathbf{\Delta}^T \mathbf{F} = (\delta\mathbf{\Delta})^T \int_V \mathbf{B}^T E \mathbf{B} dV \mathbf{\Delta} \quad \rightarrow \delta \mathbf{\Delta}^T \mathbf{F} - (\delta\mathbf{\Delta})^T \int_V \mathbf{B}^T E \mathbf{B} dV \mathbf{\Delta} = 0$$

Member stiffness matrix (4)
-Plane truss

$$\delta\mathbf{\Delta}^T\left(\mathbf{F}-\int_V \mathbf{B}^T E\mathbf{B}dV\,\mathbf{\Delta}\right)=0$$

$$\mathbf{F}-\int_V \mathbf{B}^T E\mathbf{B}dV\,\mathbf{\Delta}=\mathbf{0} \qquad \rightarrow \mathbf{F}=\left[\int_V \mathbf{B}^T E\mathbf{B}dV\right]\mathbf{\Delta}$$

Therefore, the element stiffness matrix is given as follows,

$$\mathbf{k}=\int_V \mathbf{B}^T E\mathbf{B}dV$$

Member stiffness matrix (5)
-Plane truss

$$\mathbf{k} = \int_V \mathbf{B}^T E \mathbf{B} dV$$

$$\mathbf{B} = \frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{k} = \frac{E}{L^2}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\int_V dV$$

$$\mathbf{k} = \frac{E}{L^2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\int_V dV$$

For a truss element, $\int_V dV = V = AL$

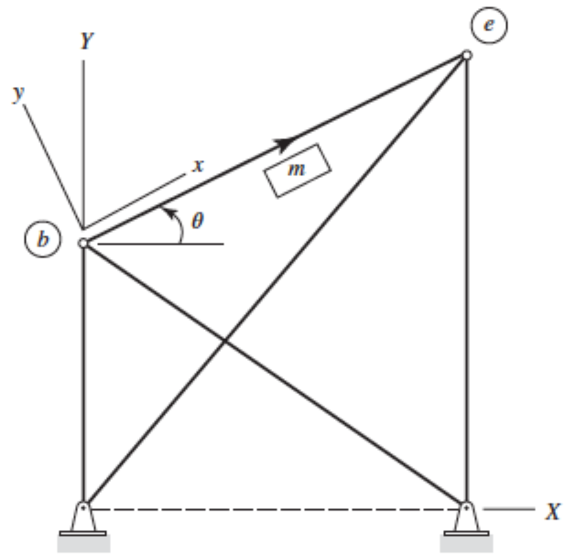
$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Member stiffness matrix (6)
-Plane truss

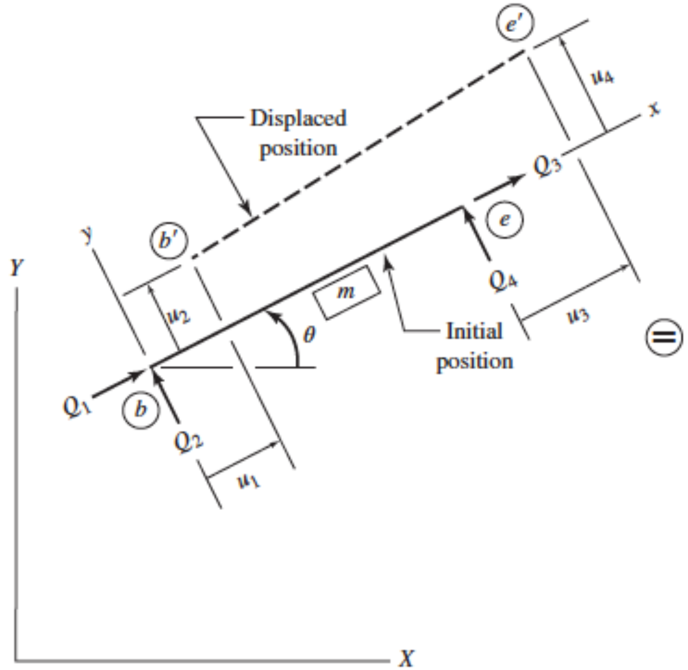
Note that the stiffness matrix is symmetric

$$\mathbf{k} = \mathbf{k}^T$$

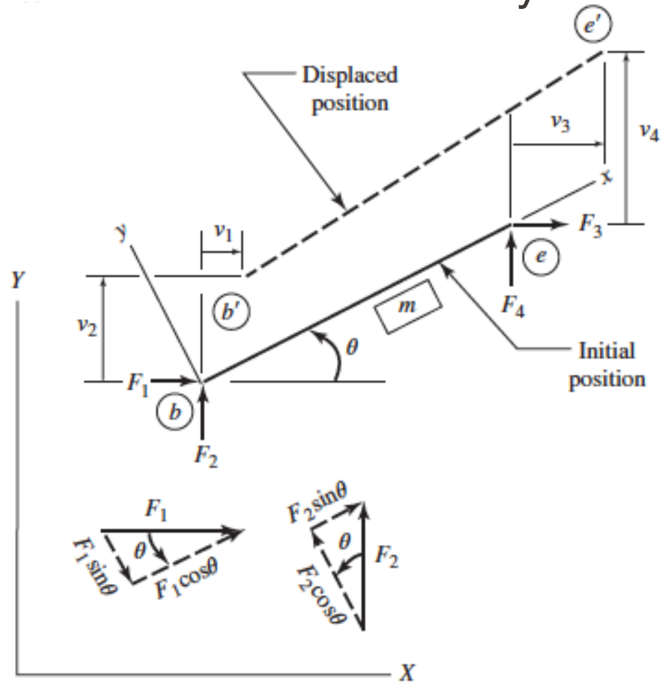
Coordinate transformations



Local coordinate system



Global coordinate system



$$Q_1 = F_1 cos\theta + F_2 sin\theta$$

$$Q_2 = -F_1 sin\theta + F_2 cos\theta$$

$$Q_3 = F_3 cos\theta + F_4 sin\theta$$

$$Q_4 = -F_3 sin\theta + F_4 cos\theta$$

Coordinate transformations (2)

Force transformation

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} cos\theta & sin\theta & 0 & 0 \\ -sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & cos\theta & sin\theta \\ 0 & 0 & -sin\theta & cos\theta \end{bmatrix}}_{\mathbf{T}\ \text{Transformation matrix}} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

$$\boldsymbol{Q} = \mathbf{T}\ \mathbf{F}$$

$$cos\theta = \frac{x_e - x_b}{L} = \frac{x_e - x_b}{\sqrt{(x_e - x_b)^2 + (y_e - y_b)^2}} \qquad sin\theta = \frac{y_e - y_b}{L} = \frac{y_e - y_b}{\sqrt{(x_e - x_b)^2 + (y_e - y_b)^2}}$$

Coordinate transformations (3)

End displacement transformation

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \underbrace{\begin{bmatrix} cos\theta & sin\theta & 0 & 0 \\ -sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & cos\theta & sin\theta \\ 0 & 0 & -sin\theta & cos\theta \end{bmatrix}}_{\mathbf{T}} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

$\mathbf{T}$ Transformation matrix

$$\mathbf{\Delta} = \mathbf{T}\ \mathbf{v}$$

Transformation from local to global coordinate systems

The transformation matrix $\mathbf{T}$ is an orthogonal matrix; thus,

$$\mathbf{T}^{-1} = \mathbf{T}^{T}$$

Forces from local to global coordinate system:

$$\mathbf{F} = \mathbf{T}^{-1}\boldsymbol{Q} = \mathbf{T}^{T}\boldsymbol{Q}$$

Displacements from local to global coordinate system:

$$\boldsymbol{v} = \mathbf{T}^{T}\,\boldsymbol{\Delta}$$

Member stiffness relations in the global coordinate system

Forces from local to global coordinate system:

$$\mathbf{F} = \mathbf{T}^T Q = \mathbf{T}^T \mathbf{k}\boldsymbol{\Delta}$$

Local stiffness matrix

$$\boldsymbol{\Delta} = (\mathbf{T}^T\mathbf{k})^{-1}\mathbf{F} = \mathbf{k}^{-1}(\mathbf{T}^T)^{-1}\mathbf{F} = \mathbf{k}^{-1}\mathbf{T}\mathbf{F}$$

Displacements from local to global coordinate system:

$$\boldsymbol{\nu} = \mathbf{T}^T\,\boldsymbol{\Delta} = \mathbf{T}^T\mathbf{k}^{-1}\mathbf{T}\mathbf{F} = (\mathbf{T}^T\mathbf{k}^{-1}\mathbf{T})\mathbf{F}$$

Member flexibility matrix in the global coordinate system

$$\mathbf{K}^{-1} = (\mathbf{T}^T\mathbf{k}^{-1}\mathbf{T})$$

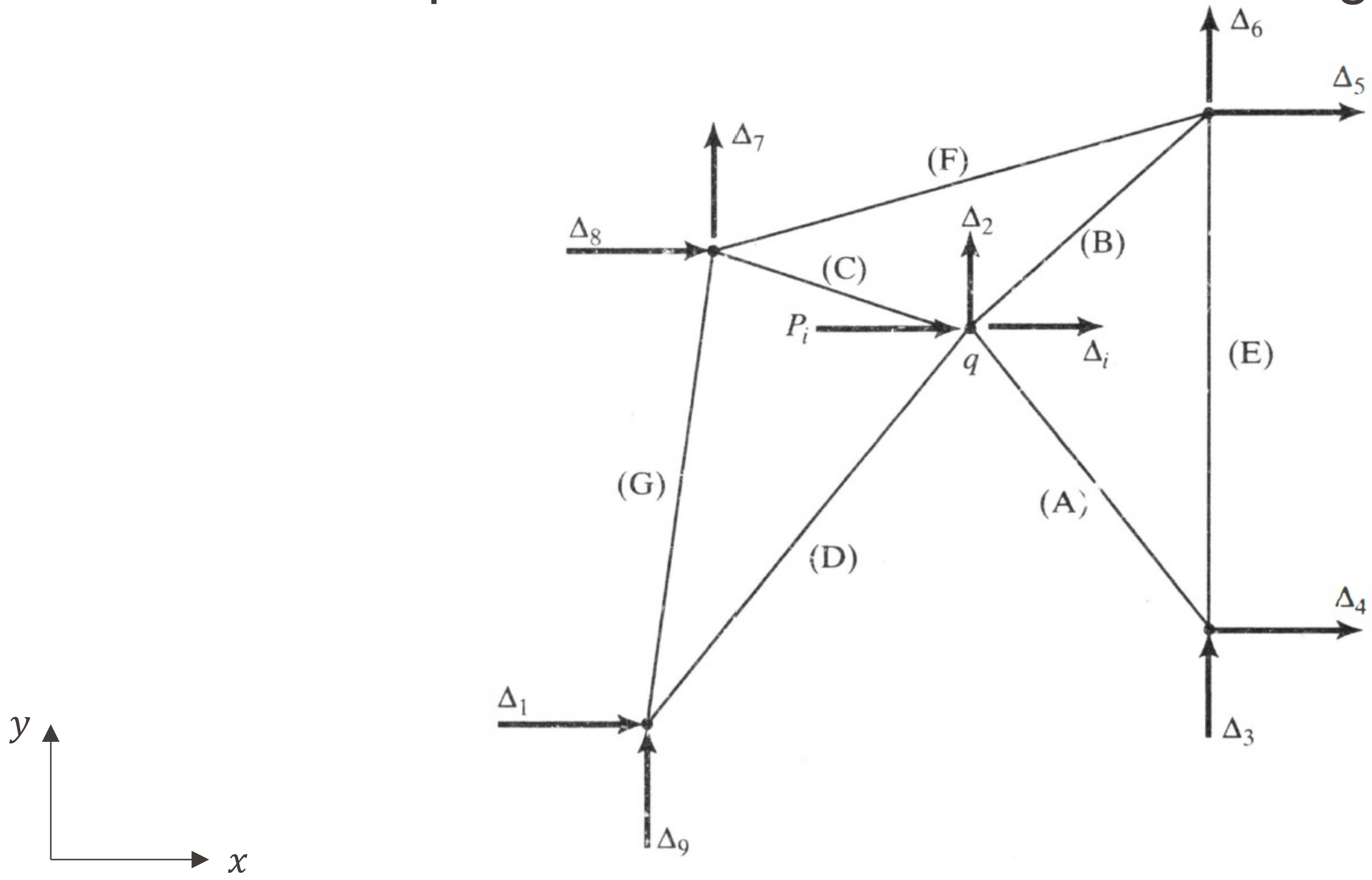
Member stiffness relations in the global coordinate system

For a planar truss element

$$\mathbf{K} = (\mathbf{T}^T\mathbf{k}\mathbf{T}) = \frac{EA}{L}\begin{bmatrix} \cos^2\phi & sin\phi cos\phi & -\cos^2\phi & -sin\phi cos\phi \\ sin\phi cos\phi & \sin^2\phi & -sin\phi cos\phi & -\sin^2\phi \\ -\cos^2\phi & -sin\phi cos\phi & \cos^2\phi & sin\phi cos\phi \\ -sin\phi cos\phi & -\sin^2\phi & sin\phi cos\phi & \sin^2\phi \end{bmatrix}$$

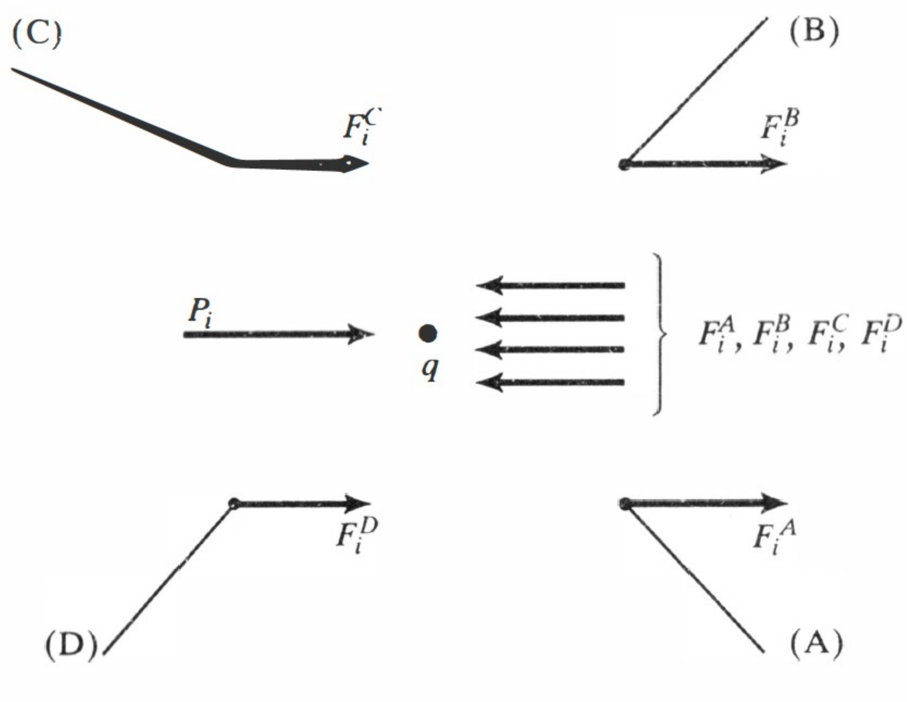
The direct stiffness method

Let us assume the plane truss below with identified degrees of freedom

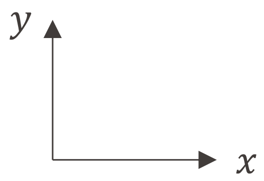


The direct stiffness method (2)

At junction q



Global x-direction internal force on bars



$$P_i = F_i^A + F_i^B + F_i^C + F_i^D$$

The direct stiffness method (3)

These forces yield expressions for $F_i^A, \dots F_i^D$ in terms of the corresponding element degrees of freedom $\Delta_i^A, \dots \Delta_9^D$; hence,

$$P_i = F_i^A + F_i^B + F_i^C + F_i^D =$$

$$\begin{aligned} P_i = {} & k_{ii}^A \Delta_i^A + k_{i2}^A \Delta_2^A + k_{i3}^A \Delta_3^A + k_{i4}^A \Delta_4^A \\ & + k_{ii}^B \Delta_i^B + k_{i2}^B \Delta_2^B + k_{i3}^B \Delta_3^B + k_{i4}^B \Delta_4^B \\ & + k_{ii}^C \Delta_i^C + k_{i2}^C \Delta_2^C + k_{i3}^C \Delta_3^C + k_{i4}^C \Delta_4^C \\ & + k_{ii}^D \Delta_i^D + k_{i2}^D \Delta_2^D + k_{i3}^D \Delta_3^D + k_{i4}^D \Delta_4^D \end{aligned}$$

The direct stiffness method (4)

At the joint the condition of displacement compatibility applies

$$\Delta_i^A = \Delta_i^B = \Delta_i^C = \Delta_i^D = \Delta_i$$

Therefore,

$$P_i = (k_{ii}^A + k_{ii}^B + k_{ii}^C + k_{ii}^D)\Delta_i + k_{i1}^D \Delta_1 \ (k_{i2}^A + k_{i2}^B + k_{i2}^C + k_{i2}^D)\Delta_2 + k_{i3}^A \Delta_3 + \ldots\ldots k_{i9}^D \Delta_9$$

or

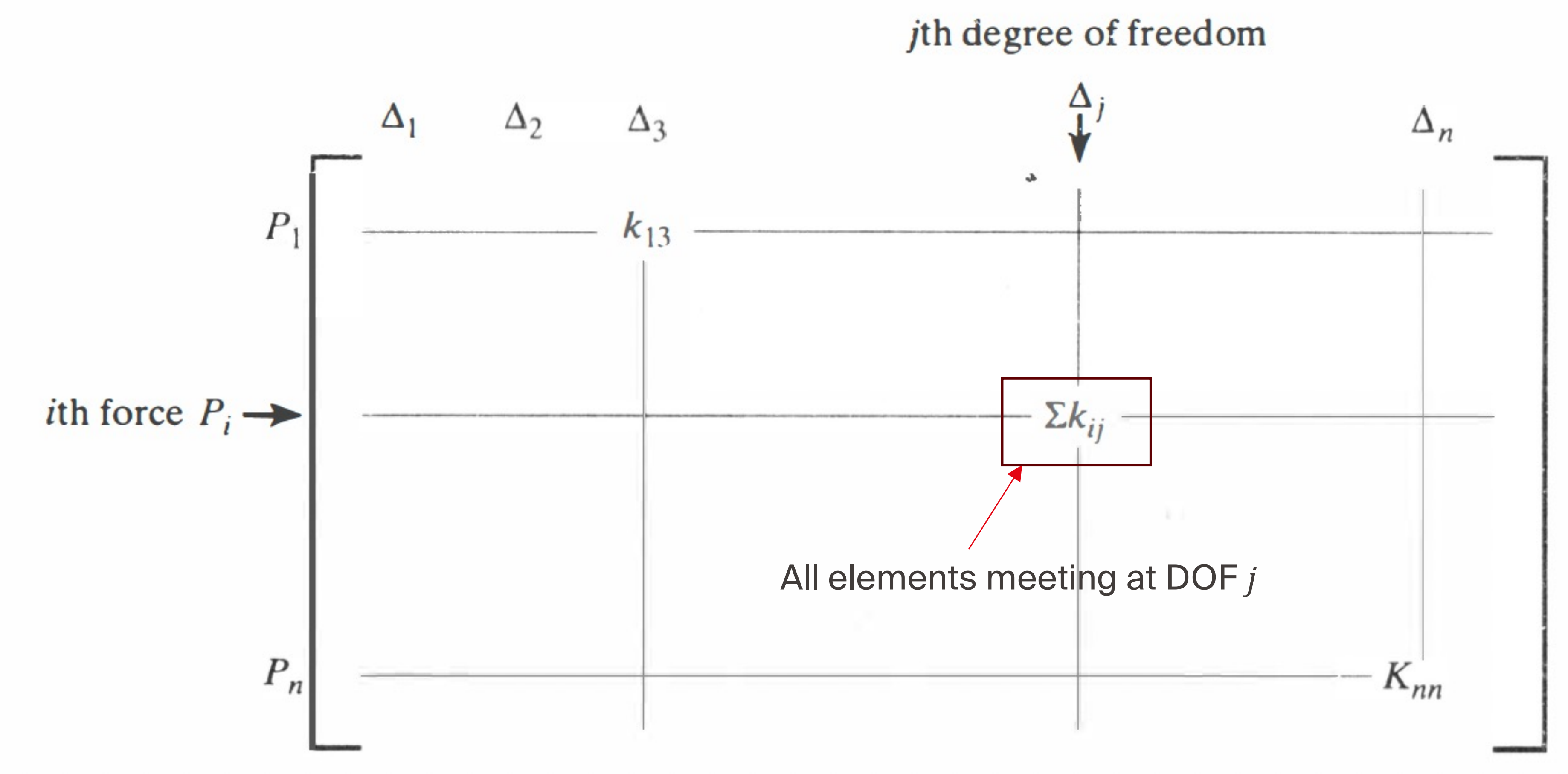
$$P_i = K_{ii}\Delta_i + K_{i1}\Delta_2 + K_{i3}\Delta_3 + \ldots\ldots K_{i9}\Delta_9$$

Global stiffness coefficients

The direct stiffness method (5)

- Each element stiffness coefficient is assigned a double subscript (k_{ij}). The first subscript in the element stiffness coefficient designates the force for which the equation is written, the second subscript designates the degree of freedom.
- Global stiffness matrix is aways square whose size is equal to the number of degrees of freedom in the complete system (see next page).
  - First subscript pertains to the force equation
  - Second subscript to the degree of freedom
- Support conditions are accounted for by noting which displacements are zero and then removing from the equations the columns or stiffness coefficients multiplying these degrees of freeedom.
- The remaining equations are solved to the global coordinate system.
- Member forces are then computed in the local coordinate system.

The direct stiffness method (6)



The direct stiffness method (7)

In a matrix form:

$$\{\mathbf{P}\} = [\mathbf{K}]\{\boldsymbol{\Delta}\}$$

Assume that the support degrees of freedom $\{\boldsymbol{\Delta}_s\}$ are grouped together

After reordering the equations,

Remaining degrees of freedom

$$\begin{Bmatrix} \mathbf{P}_f \\ \hline \mathbf{P}_s \end{Bmatrix} = \left[\begin{array}{c:c} \mathbf{K}_{ff} & \mathbf{K}_{fs} \\ \hdashline \mathbf{K}_{sf} & \mathbf{K}_{ss} \end{array}\right] \begin{Bmatrix} \boldsymbol{\Delta}_f \\ \hline \boldsymbol{\Delta}_s \end{Bmatrix}$$

supports

Note that for the supports:

$$\{\boldsymbol{\Delta}_s\} = 0$$

The direct stiffness method (8)

Therefore,

$$\{\mathbf{P}_f\} = [\mathbf{K}_{ff}]\{\boldsymbol{\Delta}_f\} \qquad \{\mathbf{P}_s\} = [\mathbf{K}_{sf}]\{\boldsymbol{\Delta}_f\}$$

To compute the displacements at all unsupported nodes:

$$\{\boldsymbol{\Delta}_f\} = \left[\mathbf{K}_{ff}\right]^{-1}\{\mathbf{P}_f\} = [\mathbf{D}]\{\mathbf{P}_f\}$$

Global flexibility matrix

To compute the support reactions:

$$\{\mathbf{P}_s\} = [\mathbf{K}_{sf}]\,[\mathbf{D}]\{\mathbf{P}_f\}$$

To obtain the internal force distribution in the i-th element

$$\{\mathbf{F}^i\} = [\mathbf{k}^i]\,\{\boldsymbol{\Delta}^i\}$$

Static condensation

The term condensation refers to the contraction in size of a system of equations by elimination of certain degrees of freedom.

$$\left\{\begin{array}{c}\mathbf{P}_c \\ \hline \mathbf{P}_b\end{array}\right\} = \left[\begin{array}{c|c}\mathbf{K}_{cc} & \mathbf{K}_{cb} \\ \hline \mathbf{K}_{bc} & \mathbf{K}_{bb}\end{array}\right] \left\{\begin{array}{c}\mathbf{\Delta}_c \\ \hline \mathbf{\Delta}_b\end{array}\right\}$$

And condenses them to the form:

$$\{\widehat{\mathbf{P}}_c\} = [\widehat{\mathbf{K}}_{cc}]\{\mathbf{\Delta}_c\}$$

First solve the lower partition and solve for $\{\mathbf{\Delta}_b\}$

$$\{\mathbf{\Delta}_b\} = [\mathbf{K}_{bb}]^{-1}\{\mathbf{P}_b\} - [\mathbf{K}_{bb}]^{-1}[\mathbf{K}_{bc}]\{\mathbf{\Delta}_c\}$$

Static condensation (2)

Substituting the previous equation into the expanded upper partition of the system:

$$\left\{\begin{array}{c}\mathbf{P}_c \\ \hline \mathbf{P}_b\end{array}\right\} = \left[\begin{array}{c|c}\mathbf{K}_{cc} & \mathbf{K}_{cb} \\ \hline \mathbf{K}_{bc} & \mathbf{K}_{bb}\end{array}\right]\left\{\begin{array}{c}\boldsymbol{\Delta}_c \\ \hline \boldsymbol{\Delta}_b\end{array}\right\}$$

$$\{\boldsymbol{P}_c\} = [\mathbf{K}_{cc}]\{\boldsymbol{\Delta}_c\} + [\mathbf{K}_{cb}]\,\{\boldsymbol{\Delta}_b\}$$

$$\{\boldsymbol{P}_c\} = [\mathbf{K}_{cc}]\{\boldsymbol{\Delta}_c\} - [\mathbf{K}_{cb}][\mathbf{K}_{bb}]^{-1}[\mathbf{K}_{bc}]\,\{\boldsymbol{\Delta}_c\} + [\mathbf{K}_{cb}][\mathbf{K}_{bb}]^{-1}\,\{\mathbf{P}_b\}$$

$$\underbrace{\{\boldsymbol{P}_c\} - [\mathbf{K}_{cb}][\mathbf{K}_{bb}]^{-1}\,\{\mathbf{P}_b\}}_{\{\widehat{\mathbf{P}}_c\}} = \underbrace{\left([\mathbf{K}_{cc}] - [\mathbf{K}_{cb}][\mathbf{K}_{bb}]^{-1}[\mathbf{K}_{bc}]\right)}_{[\widehat{\mathbf{K}}_{cc}]}\,\{\boldsymbol{\Delta}_c\}$$

Static condensation (3)

Therefore, the unknown displacements can be calculated as follows:

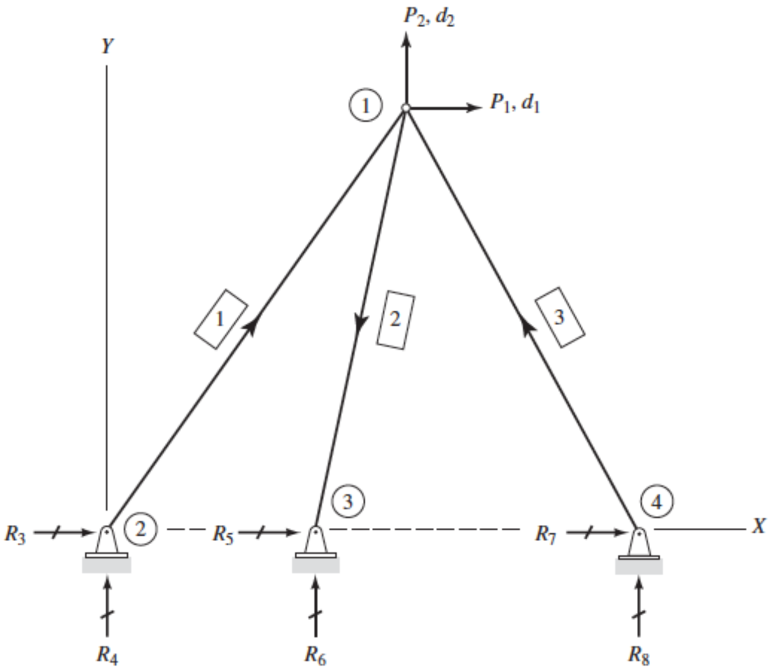
$$\{\boldsymbol{\Delta}_c\} = [\widehat{\mathbf{K}}_{cc}]^{-1}\{\widehat{\mathbf{P}}_c\}$$

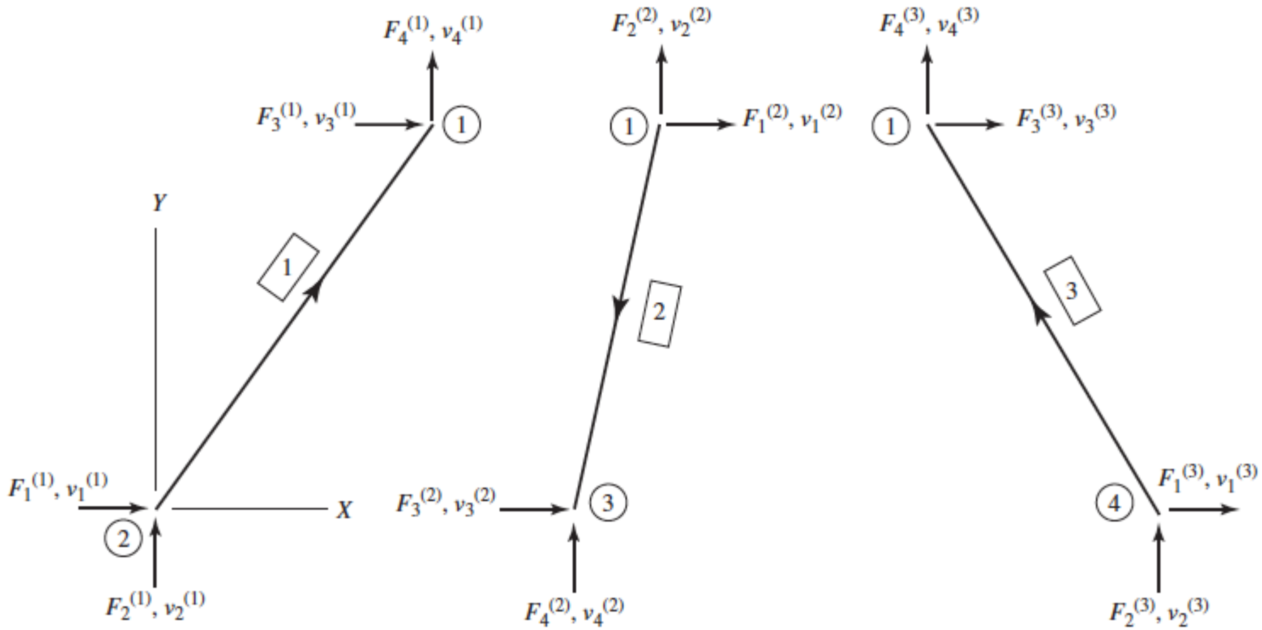
And finally,

$$\{\boldsymbol{\Delta}_b\} = [\mathbf{K}_{bb}]^{-1}\,\{\mathbf{P}_b\} - [\mathbf{K}_{bb}]^{-1}[\mathbf{K}_{bc}]\,\{\boldsymbol{\Delta}_{\boldsymbol{c}}\}$$

The inversion of matrices $[\mathbf{K}_{bb}]$ and $[\widehat{\mathbf{K}}_{cc}]^{-1}$ are easier to handle than the original global stiffness matrix $[\mathbf{K}]$, which is usually ill-conditioned.

Physical interpretation of global stiffness matrix

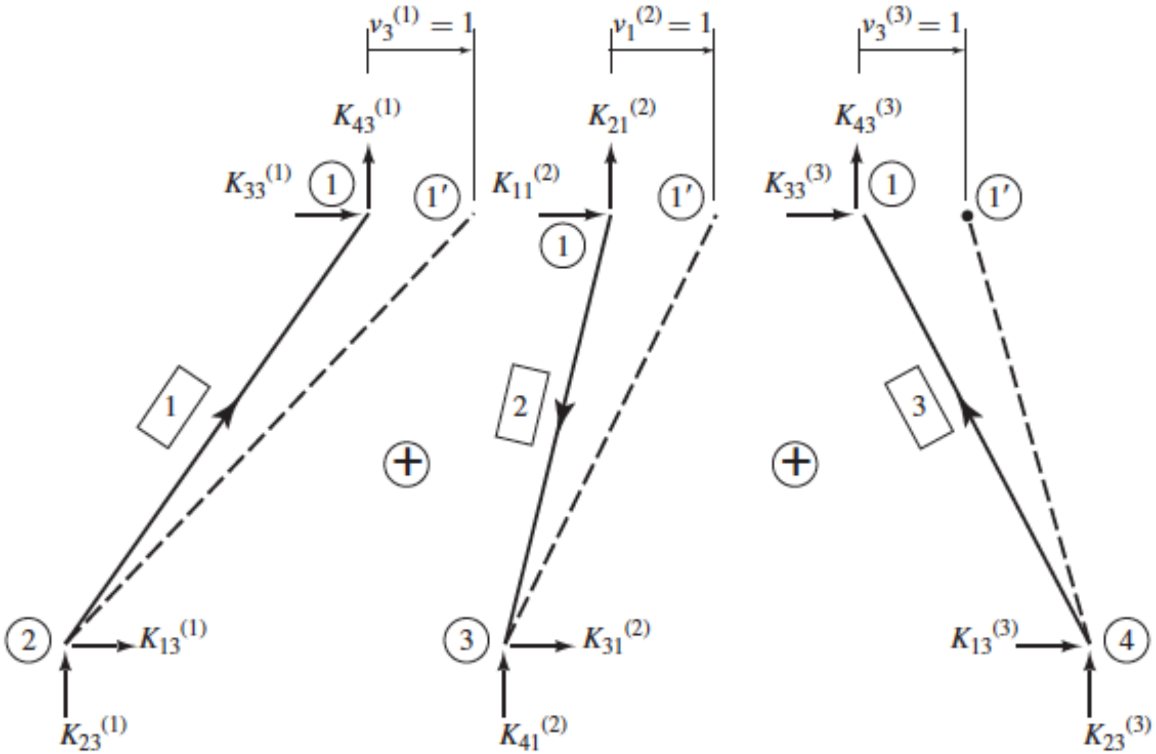




$$\mathbf{P} = \mathbf{S}\mathbf{d}$$ (After static condensation)

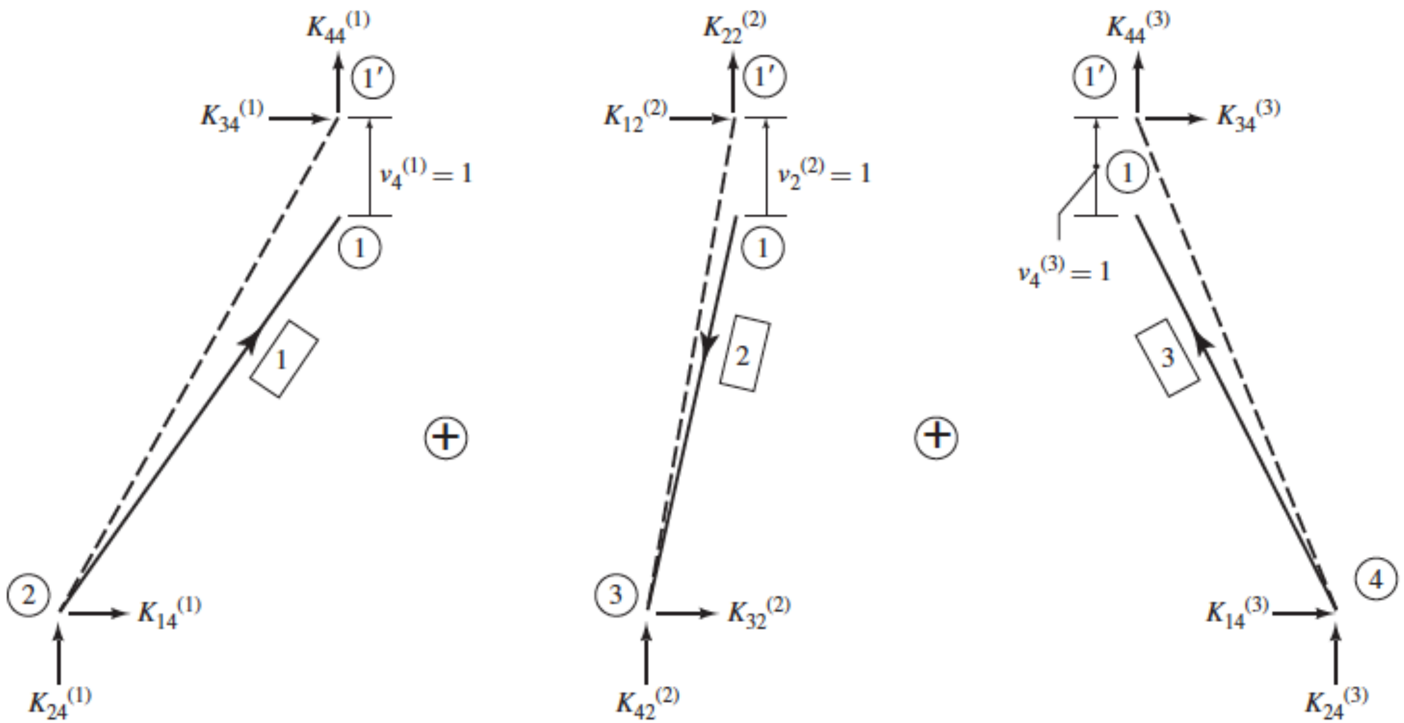
$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

Physical interpretation of global stiffness matrix (2)



$$S_{11} = K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)}$$

Physical interpretation of global stiffness matrix (3)



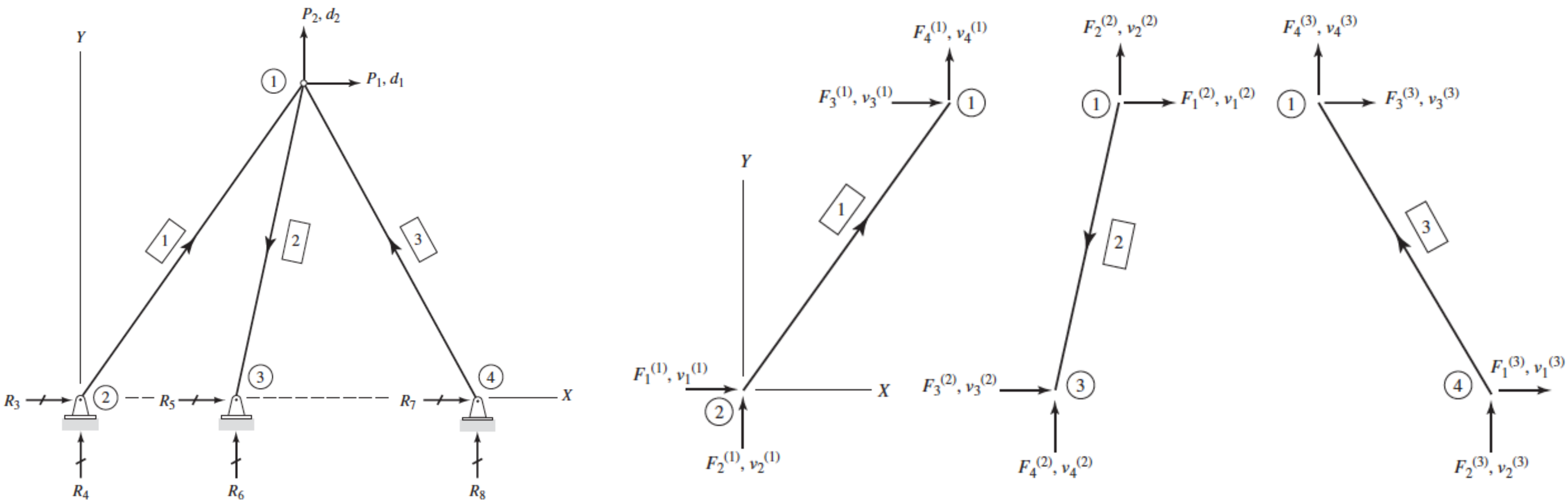
$$S_{12} = K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)}$$

Assembly of the structure stiffness matrix
-Code number technique

- The structure stiffness coefficient of a joint in a direction equals the algebraic sum of the member stiffness coefficients, in that direction, at all the member ends connected to the joint.

- The structure stiffness matrix can be formulated directly by adding the elements of the member stiffness matrices into their proper positions in the structure matrix.

Assembly of the structure stiffness matrix
-Code number technique



Assembly of the structure stiffness matrix (2)

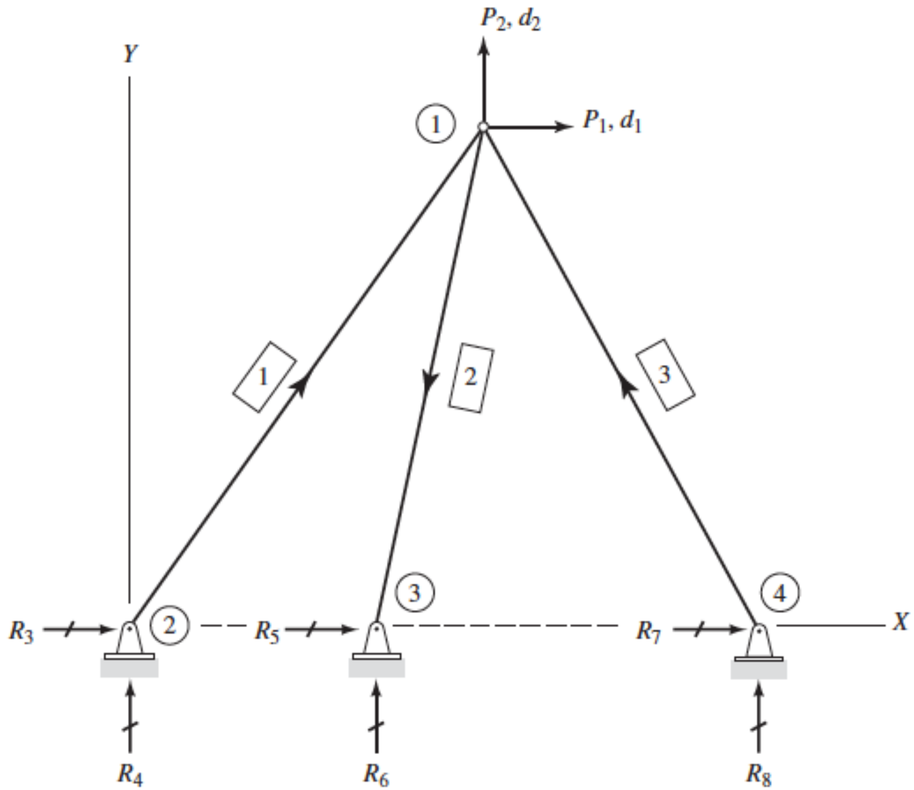
-Code number technique

$$
\mathbf{K}_1 = \begin{array}{c}
\begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\
\left[\begin{array}{cccc}
K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & K_{14}^{(1)} \\
K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} \\
K_{31}^{(1)} & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} \\
K_{41}^{(1)} & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)}
\end{array}\right]
\end{array}
\begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array}
\qquad
\mathbf{K}_2 = \begin{array}{c}
\begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\
\left[\begin{array}{cccc}
K_{11}^{(2)} & K_{12}^{(2)} & K_{13}^{(2)} & K_{14}^{(2)} \\
K_{21}^{(2)} & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} \\
K_{31}^{(2)} & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} \\
K_{41}^{(2)} & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)}
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}
\qquad
\mathbf{K}_3 = \begin{array}{c}
\begin{array}{cccc} 7 & 8 & 1 & 2 \end{array} \\
\left[\begin{array}{cccc}
K_{11}^{(3)} & K_{12}^{(3)} & K_{13}^{(3)} & K_{14}^{(3)} \\
K_{21}^{(3)} & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} \\
K_{31}^{(3)} & K_{32}^{(3)} & K_{33}^{(3)} & K_{34}^{(3)} \\
K_{41}^{(3)} & K_{42}^{(3)} & K_{43}^{(3)} & K_{44}^{(3)}
\end{array}\right]
\end{array}
\begin{array}{c} \\ 7 \\ 8 \\ 1 \\ 2 \end{array}
$$

$$
\mathbf{S} = \begin{array}{c}
\begin{array}{cc} 1 & 2 \end{array} \\
\left[\begin{array}{cc}
K_{33}^{(1)} + K_{11}^{(2)} + K_{33}^{(3)} & K_{34}^{(1)} + K_{12}^{(2)} + K_{34}^{(3)} \\
K_{43}^{(1)} + K_{21}^{(2)} + K_{43}^{(3)} & K_{44}^{(1)} + K_{22}^{(2)} + K_{44}^{(3)}
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \end{array}
$$

Assembly of support reaction vector

-Code number technique



$$\mathbf{F}_1 = \begin{array}{c} \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ \hdashline F_3^{(1)} \\ F_4^{(1)} \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array}$$

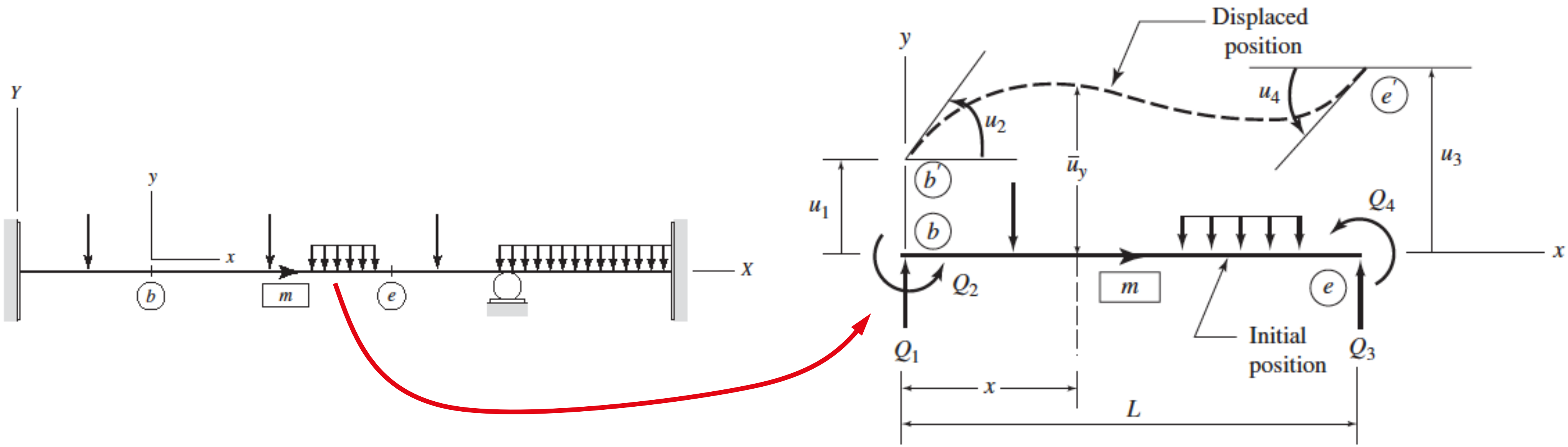
$$\mathbf{R} = \begin{bmatrix} R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{bmatrix} = \begin{bmatrix} F_1^{(1)} \\ F_2^{(1)} \\ F_3^{(2)} \\ F_4^{(2)} \\ F_1^{(3)} \\ F_2^{(3)} \end{bmatrix} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

$$\mathbf{F}_2 = \begin{bmatrix} F_1^{(2)} \\ F_2^{(2)} \\ \hdashline F_3^{(2)} \\ F_4^{(2)} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$\mathbf{F}_3 = \begin{bmatrix} F_1^{(3)} \\ F_2^{(3)} \\ \hdashline F_3^{(3)} \\ F_4^{(3)} \end{bmatrix} \begin{array}{c} 7 \\ 8 \\ 1 \\ 2 \end{array}$$

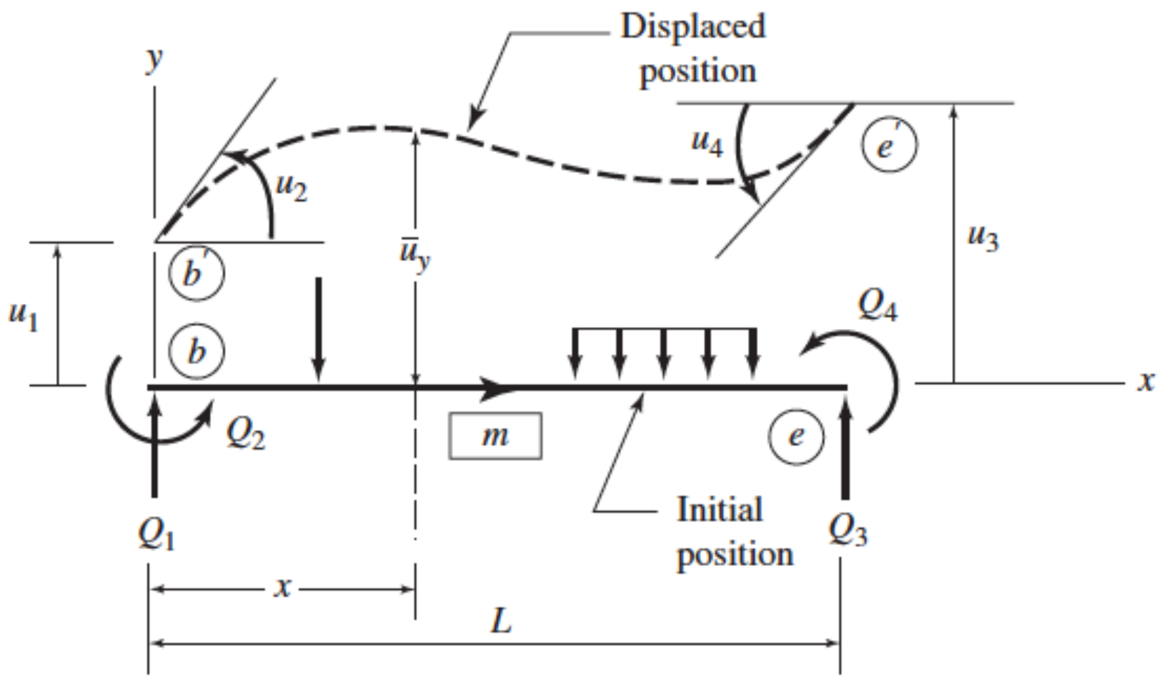
Elastic beam-column elements

We will deal with prismatic members (constant axial and flexural rigidities, EA, EI, respectively)



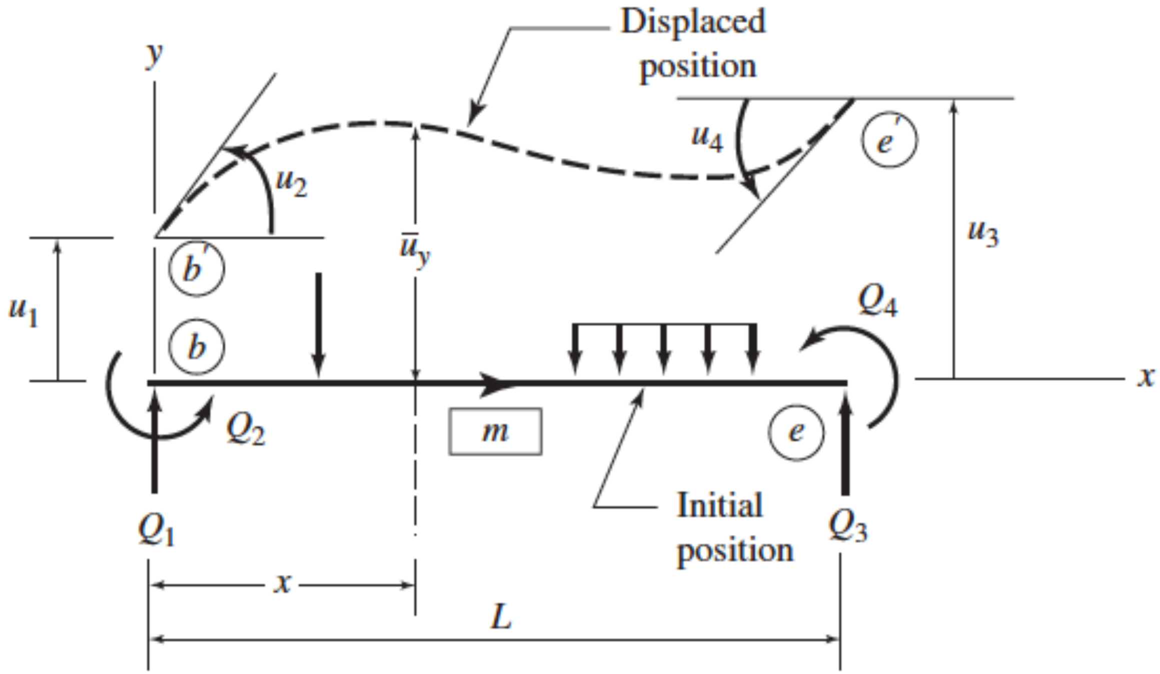
Elastic beam-column elements

When a beam is subjected to external loads, internal moments and shears generally develop at the ends of its individual members.



Elastic beam-column elements (2)

Two displacements (y-direction) and rotation (about the z-axis) are necessary to specify the displaced position of each end of the member.

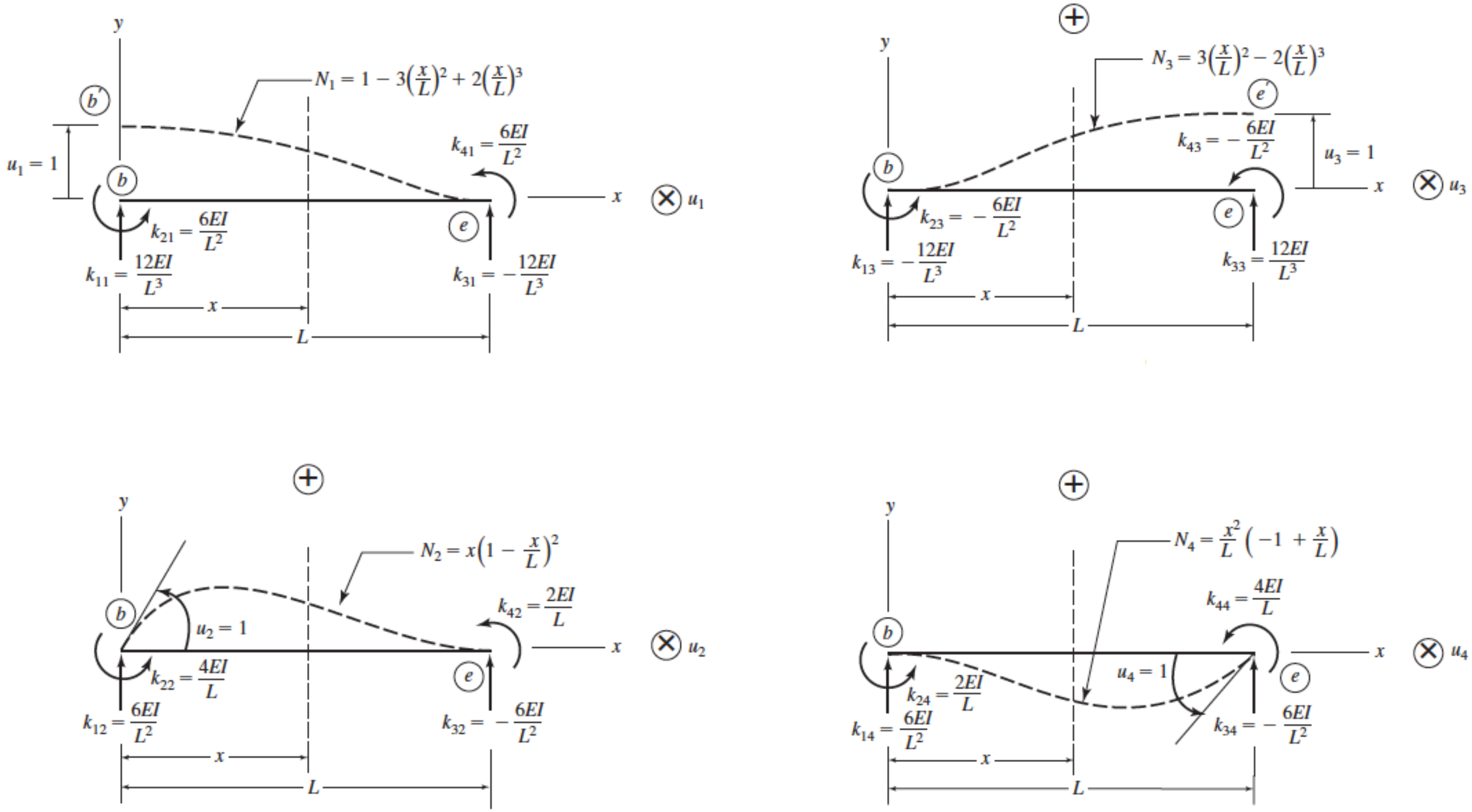


Elastic beam-column elements (3)

It is convenient to establish the member end forces and displacements as the algebraic sum of the end forces required to cause the individual end displacements and the forces caused by the external loads acting on the member with no end displacements (fixed-end).

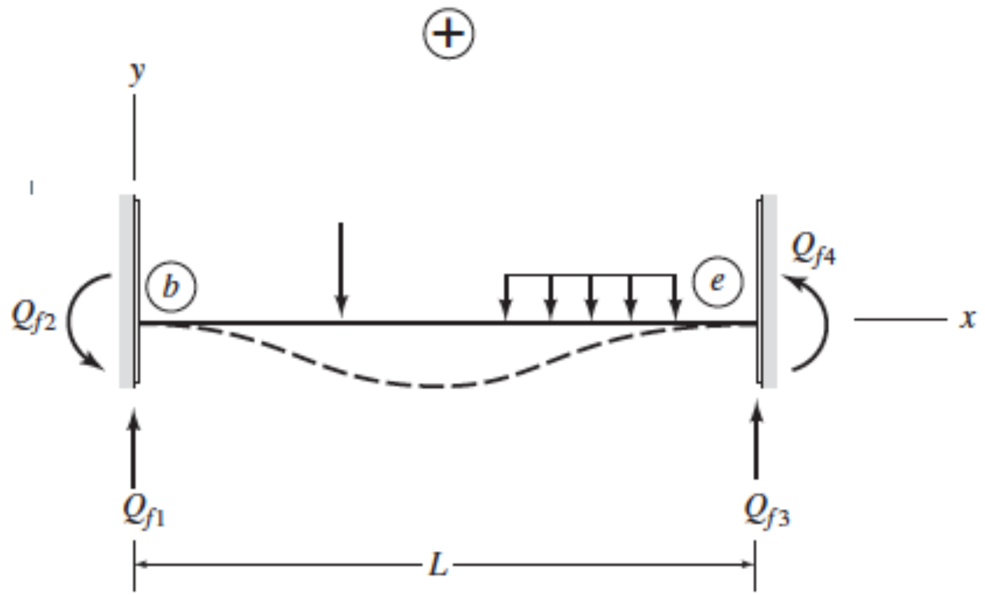
$$\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$$

Elastic beam-column elements (4)

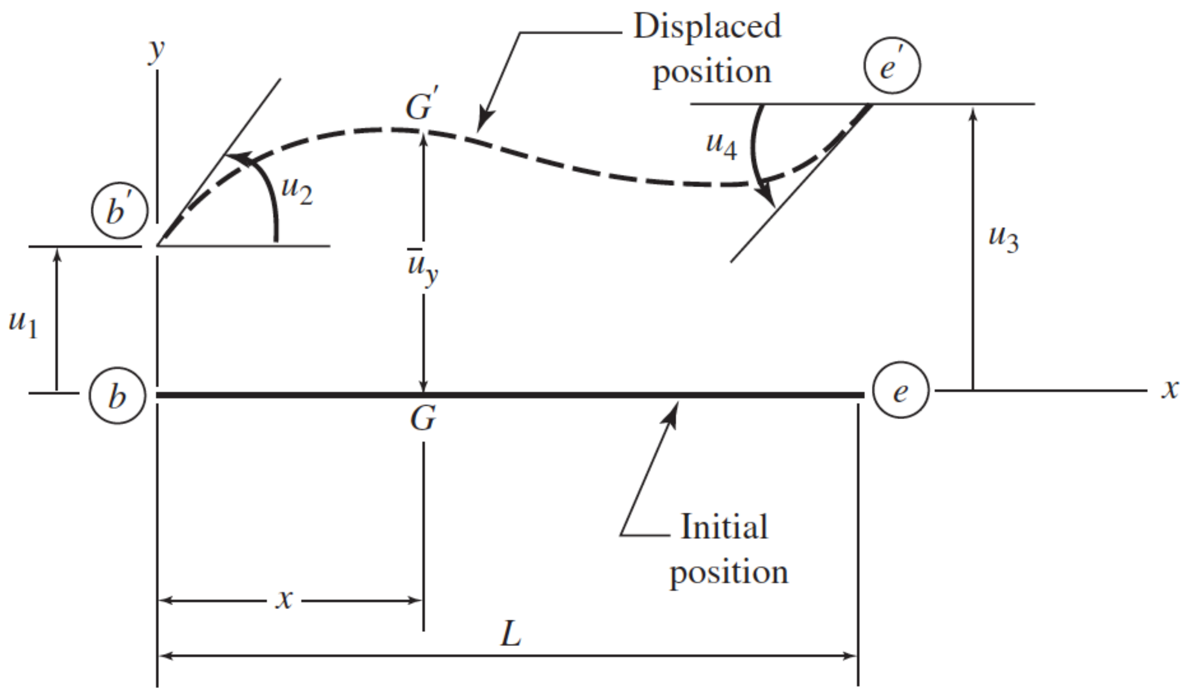


Elastic beam-column elements (5)

-Fixed end forces due to loads



Member forces and displacements
-Local coordinate system



End b

$$x = 0, \qquad \bar{u}_y = u_1$$

$$x = 0, \qquad \theta = u_2$$

End e

$$x = L, \qquad \bar{u}_y = u_3$$

$$x = L, \qquad \theta = u_4$$

Recall that, the slope at any point, $\theta = \dfrac{d\bar{u}_y(x)}{dx}$

Member forces and displacements
-Local coordinate system

With four boundary conditions we can use a cubic interpolation function for the displacement function $\bar{u}_y$ to approximate the displacements within the member as a function of the end displacements,

$$\bar{u}_y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

The slope of the member at any position, x within the member,

$$\theta(x) = \frac{d\bar{u}_y(x)}{dx} = a_1 + 2a_2 x + 3a_3 x^2$$

Member forces and displacements
-Local coordinate system

At $x = 0$,

$$\bar{u}_y(0) = a_0 = u_1$$

$$\theta(0) = a_1 = u_2$$

At $x = L$,

$$\bar{u}_y(L) = a_0 + a_1 L + a_2 L^2 + a_3 L^3 = u_3$$

$$\theta(L) = a_1 + 2a_2 L + 3a_3 L^2 = u_4$$

$$a_0 = u_1$$

$$a_1 = u_2$$

$$a_2 = \frac{1}{L^2}(-3u_1 - 2u_2 L + 3u_3 - u_4 L)$$

$$a_3 = \frac{1}{L^3}(2u_1 + u_2 L - 2u_3 + u_4 L)$$

Member forces and displacements
-Local coordinate system

Finally,

$$\bar{u}_y(x) = \left[1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right] u_1 + \left[x\left(1 - \frac{x}{L}\right)^2\right] u_2$$
$$+ \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right] u_3 + \left[\frac{x^2}{L}\left(-1 + \frac{x}{L}\right)\right] u_4$$

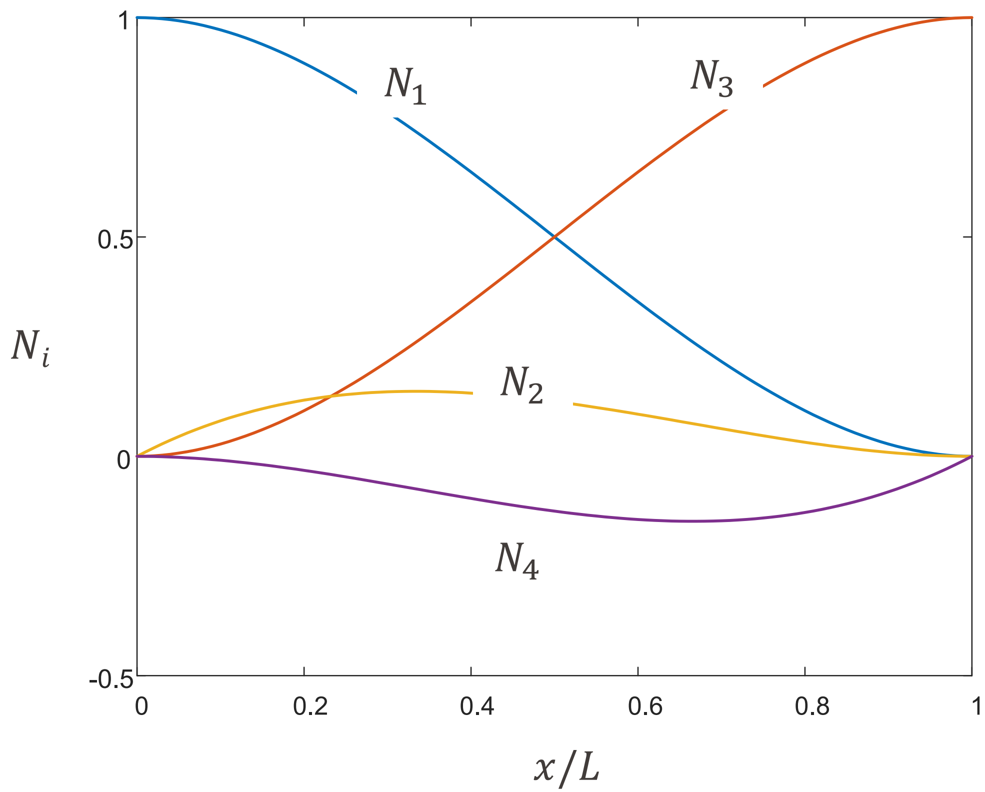
Shape functions

$$N_1 = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

$$N_2 = \left[x\left(1 - \frac{x}{L}\right)^2\right]$$

$$N_3 = \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right]$$

$$N_4 = \left[\frac{x^2}{L}\left(-1 + \frac{x}{L}\right)\right]$$



Displacement field in a matrix form

$$\bar{u}_y(x) = \left[1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right]u_1 + \left[x\left(1 - \frac{x}{L}\right)^2\right]u_2 + \left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3\right]u_3 + \left[\frac{x^2}{L}\left(-1 + \frac{x}{L}\right)\right]u_4$$

$$\bar{u}_y(x) = \underbrace{\begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}}_{\mathbf{N}} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\bar{u}_y(x) = \mathbf{N}\boldsymbol{\Delta}$$

Strain-displacement relationship

Recall from structural mechanics,

The normal (longitudinal) strain in a fiber of a member located at a distance y above the neutral axis can be expressed as follows,

$$\varepsilon(x) = -y\frac{d^2\bar{u}_y}{dx^2} = -y\frac{d^2}{dx^2}(\mathbf{N\Delta}) = \left(-y\frac{d^2\mathbf{N}}{dx^2}\right)\mathbf{\Delta} = \mathbf{B\Delta}$$

Member strain-displacement matrix, $\mathbf{B}$

$$\mathbf{B} = -\mathrm{y}\left[\frac{d^2N_1}{dx^2} \quad \frac{d^2N_2}{dx^2} \quad \frac{d^2N_3}{dx^2} \quad \frac{d^2N_4}{dx^2}\right]$$

Strain-displacement relationship (2)

For a typical beam-column element

$$\mathbf{B} = -\frac{y}{L^2}\left[6\left(-1+2\frac{x}{L}\right) \quad 2L\left(-2+3\frac{x}{L}\right) \quad 6\left(1-2\frac{x}{L}\right) \quad 2L\left(-1+3\frac{x}{L}\right)\right]$$

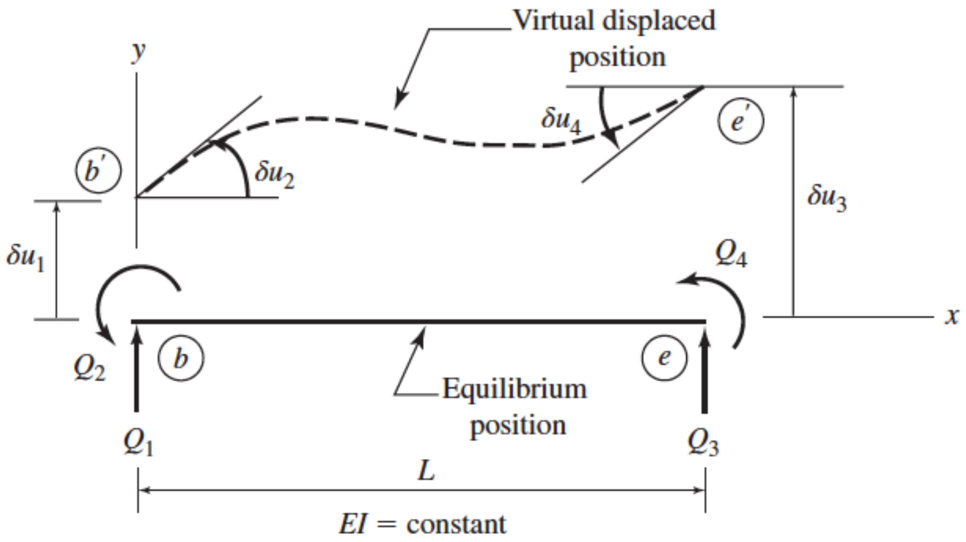
Stress-displacement relationship

The stress-strain relation is as follows,

$$\sigma = E\varepsilon = E\mathbf{B}\boldsymbol{\Delta}$$

Member stiffness matrix

We can establish the relationship between member end forces $\mathbf{F}$ and end displacements, by applying the principle of virtual work for deformable bodies



$$\delta W_e = Q_1 \delta u_1 + Q_2 \delta u_2 + Q_3 \delta u_3 + Q_4 \delta u_4$$

$$\delta W_e = [\delta u_1 \quad \delta u_2 \quad \delta u_3 \quad \delta u_4] \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$$

$$\delta W_e = \delta \mathbf{\Delta}^T \mathbf{F}$$

Member stiffness matrix (2)

for the planar beam-column element

$$\delta W_e = \delta \mathbf{\Delta}^T \mathbf{F} = \int_V \delta \varepsilon^T \sigma dV$$

$$\delta W_e = \delta \mathbf{\Delta}^T \mathbf{F} = \int_V (\mathbf{B} \delta \mathbf{\Delta})^T E \mathbf{B} dV \mathbf{\Delta} \qquad (\mathbf{B} \delta \mathbf{\Delta})^T = (\delta \mathbf{\Delta})^T \mathbf{B}^T$$

$$\delta \mathbf{\Delta}^T \mathbf{F} = (\delta \mathbf{\Delta})^T \int_V \mathbf{B}^T E \mathbf{B} dV \mathbf{\Delta} \quad \rightarrow \delta \mathbf{\Delta}^T \mathbf{F} - (\delta \mathbf{\Delta})^T \int_V \mathbf{B}^T E \mathbf{B} dV \mathbf{\Delta} = 0$$

Member stiffness matrix (3)

$$\delta\boldsymbol{\Delta}^T\left(\mathbf{F}-\int_V \mathbf{B}^T E\mathbf{B}dV\,\boldsymbol{\Delta}\right)=0$$

$$\mathbf{F}-\int_V \mathbf{B}^T E\mathbf{B}dV\,\boldsymbol{\Delta}=\mathbf{0} \qquad \rightarrow \mathbf{F}=\left[\int_V \mathbf{B}^T E\mathbf{B}dV\right]\boldsymbol{\Delta}$$

Therefore, the element stiffness matrix is given as follows,

$$\mathbf{k}=\int_V \mathbf{B}^T E\mathbf{B}dV$$

Member stiffness matrix (4)

$$\mathbf{k} = \int_V \mathbf{B}^T E \mathbf{B} dV$$

$$[\mathbf{k}] = \frac{E}{L^4} \int_V y^2 \begin{Bmatrix} 6\left(-1 + \frac{2x}{L}\right) \\ 2L\left(-2 + \frac{3x}{L}\right) \\ 6\left(1 - \frac{2x}{L}\right) \\ 2L\left(-1 + \frac{3x}{L}\right) \end{Bmatrix} \left[6\left(-1 + \frac{2x}{L}\right) \quad 2L\left(-2 + \frac{3x}{L}\right) \quad 6\left(1 - \frac{2x}{L}\right) \quad 2L\left(-1 + \frac{3x}{L}\right)\right] dV$$

$$dV = (dA)dx \qquad \int_A y^2 dA = I$$

Member stiffness matrix (5)

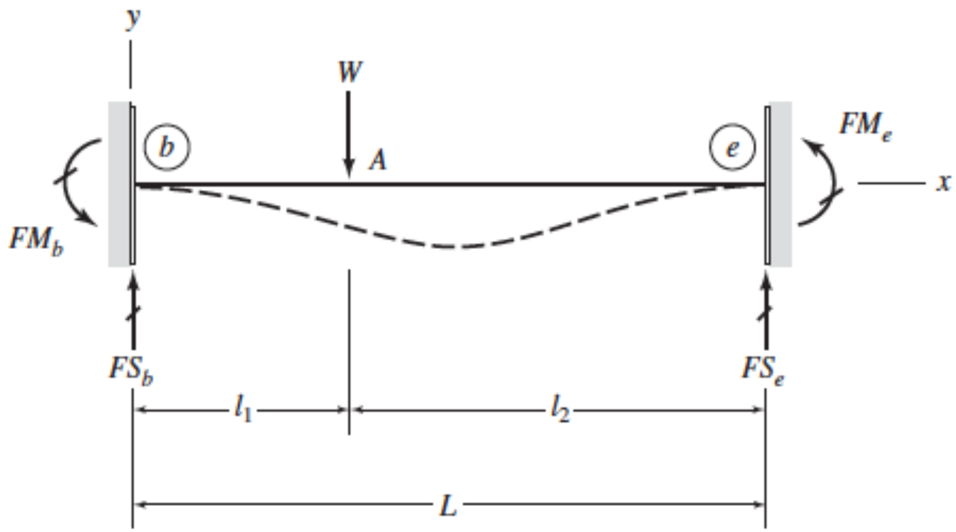
$$
\mathbf{k} = \frac{EI}{L^4}\int_0^L
$$

$$
\times \begin{bmatrix}
36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-2+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) & -36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-1+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) \\
12L\left(-2+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) & 4L^2\left(-2+3\frac{x}{L}\right)^2 & 12L\left(-2+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) & 4L^2\left(-2+3\frac{x}{L}\right)\left(-1+3\frac{x}{L}\right) \\
-36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-2+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) & 36\left(-1+2\frac{x}{L}\right)^2 & 12L\left(-1+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) \\
12L\left(-1+3\frac{x}{L}\right)\left(-1+2\frac{x}{L}\right) & 4L^2\left(-2+3\frac{x}{L}\right)\left(-1+3\frac{x}{L}\right) & 12L\left(-1+3\frac{x}{L}\right)\left(1-2\frac{x}{L}\right) & 4L^2\left(-1+3\frac{x}{L}\right)^2
\end{bmatrix} dx
$$

Member stiffness matrix (6)

$$[\mathbf{k}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Member fixed-end forces due to loads



$0 \leq x \leq l_1$ $\quad M = -FM_b + FS_b x \quad$ $$\frac{d^2\bar{u}_y}{dx^2} = \frac{1}{EI}(-FM_b + FS_b x)$$

$l_1 \leq x \leq L$ $\quad M = -FM_b + FS_b x - W(x - l_1) \quad$ $$\frac{d^2\bar{u}_y}{dx^2} = \frac{1}{EI}\left(-FM_b + FS_b x - W(x - l_1)\right)$$

Member fixed-end forces due to loads (2)

$0 \leq x \leq l_1$

$$\theta = \frac{d\bar{u}_y}{dx} = \frac{1}{EI}\left(-FM_b x + \frac{FS_b x^2}{2}\right) + C_1$$

$$\bar{u}_y = \frac{1}{EI}\left(-\frac{FM_b x^2}{2} + \frac{FS_b x^3}{6}\right) + C_1 x + C_2$$

$l_1 \leq x \leq L$

$$\theta = \frac{d\bar{u}_y}{dx} = \frac{1}{EI}\left(-FM_b x + \frac{FS_b x^2}{2} - \frac{Wx}{2}(x - 2l_1)\right) + C_3$$

$$\bar{u}_y = \frac{1}{EI}\left(-\frac{FM_b x^2}{2} + \frac{FS_b x^3}{6} - \frac{Wx^2}{6}(x - 3l_1)\right) + C_3 x + C_4$$

Member fixed-end forces due to loads (3)

$$x = 0, \qquad \theta = \bar{u}_y = 0, \qquad C_1 = C_2 = 0$$

$$x = l_1, \qquad \theta_A = \theta_B$$

$$\frac{1}{EI}\left(-FM_b l_1 + \frac{FS_b l_1^2}{2}\right) = \frac{1}{EI}\left(-FM_b l_1 + \frac{FS_b l_1^2}{2} + \frac{W l_1^2}{2}\right) + C_3$$

$$C_3 = \frac{W l_1^2}{2EI}$$

Member fixed-end forces due to loads (4)

$x = l_1,$ $\bar{u}_{y,A} = \bar{u}_{y,B}$

$$\frac{1}{EI}\left(-\frac{FM_b l_1^2}{2} + \frac{FS_b l_1^3}{2}\right) = \frac{1}{EI}\left(-\frac{FM_b l_1^2}{2} + \frac{FS_b l_1^3}{6} + \frac{W l_1^3}{2EI}\right) + C_4$$

$$C_4 = \frac{W l_1^3}{6EI}$$

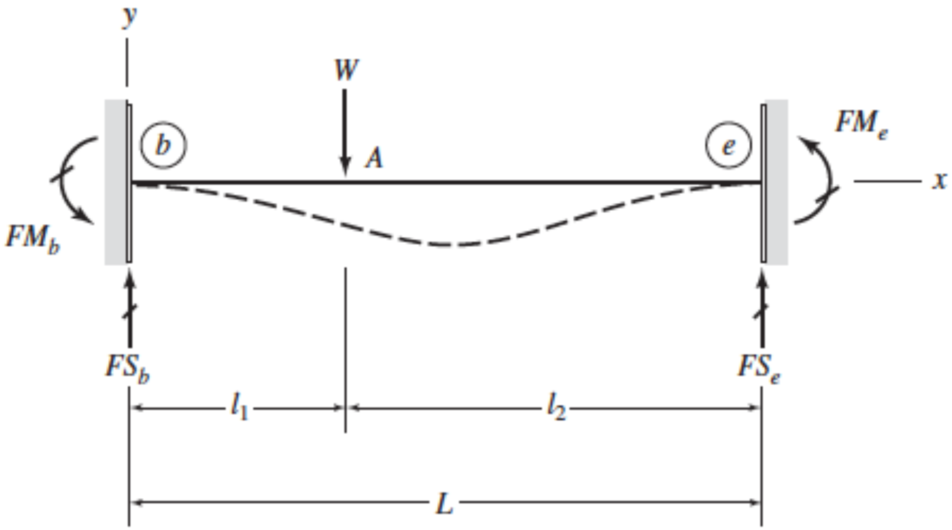
Member fixed-end forces due to loads (5)

$$x = L, \qquad \theta = \bar{u}_y = 0,$$

$$\frac{1}{EI}\left[-FM_bL + \frac{FS_bL^2}{2} - \frac{WL}{2}(L - 2l_1)\right] - \frac{Wl_1^2}{2EI} = 0$$

$$\frac{1}{EI}\left[-\frac{FM_bL^2}{2} + \frac{FS_bL^3}{6} - \frac{WL^2}{6}(L - 3l_1)\right] - \frac{Wl_1^2L}{2EI} + \frac{Wl_1^3}{6EI} = 0$$

Member fixed-end forces due to loads (6)

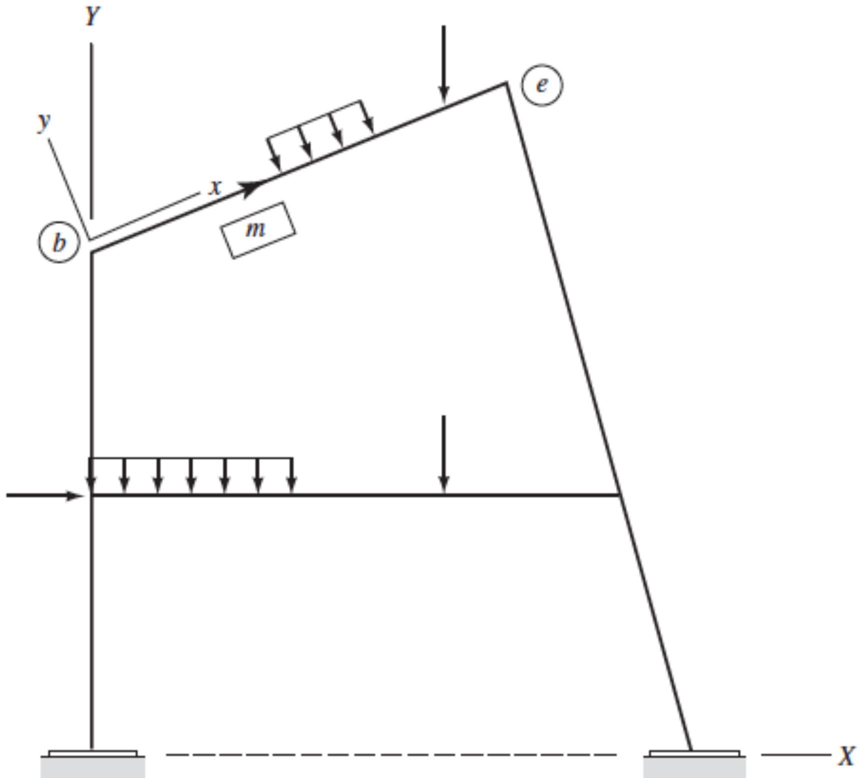


$$Q_f = \begin{Bmatrix} FS_b \\ FM_b \\ FS_e \\ FM_b \end{Bmatrix}$$

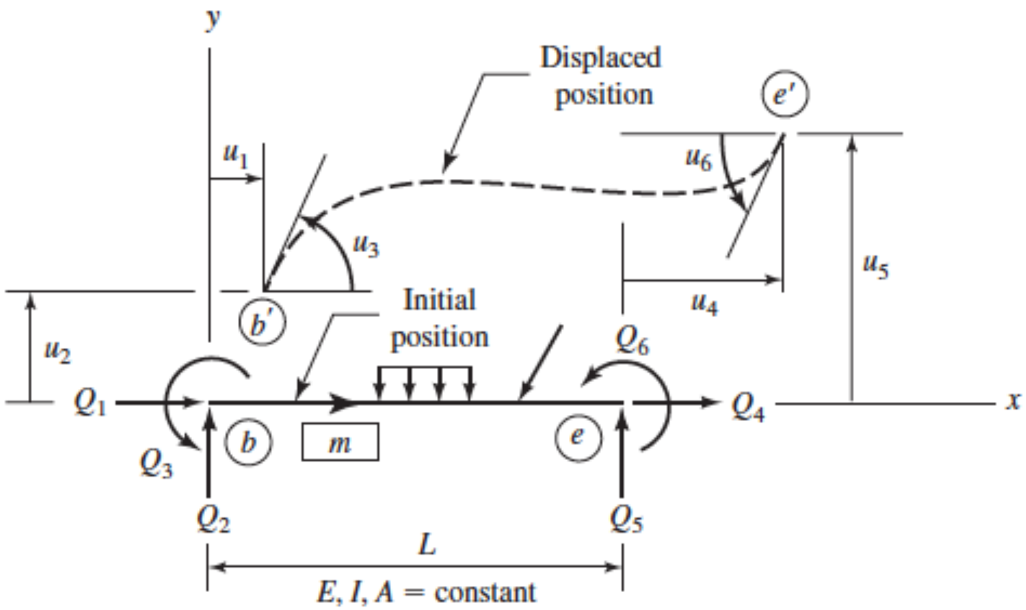
$$FS_b = \frac{W l_2^2}{L^3}(3l_1 + l_2) \qquad FM_b = \frac{W l_1 l_2^2}{L^2}$$

$$FS_e = \frac{W l_1^2}{L^3}(l_1 + 3l_2) \qquad FM_e = -\frac{W l_1^2 l_2}{L^2}$$

Coordinate transformations



Local coordinate system



$$Q_i = \sum_{j=1}^{6} (k_{ij} u_j) + Q_{fi}, \qquad i = 1,2, \dots, 6$$

Coordinate transformations (2)

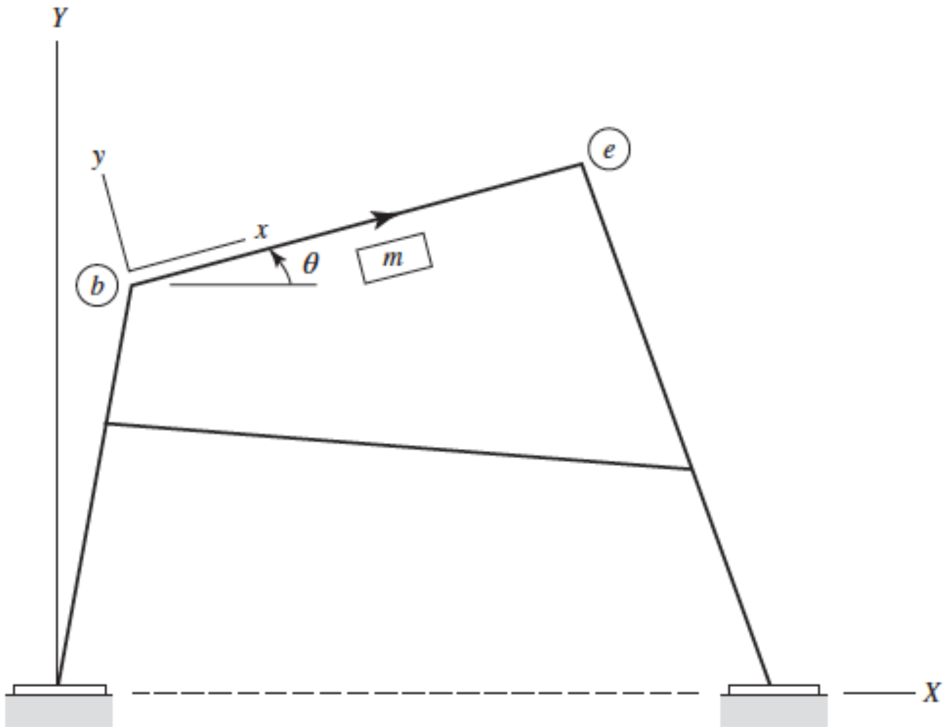
As such,

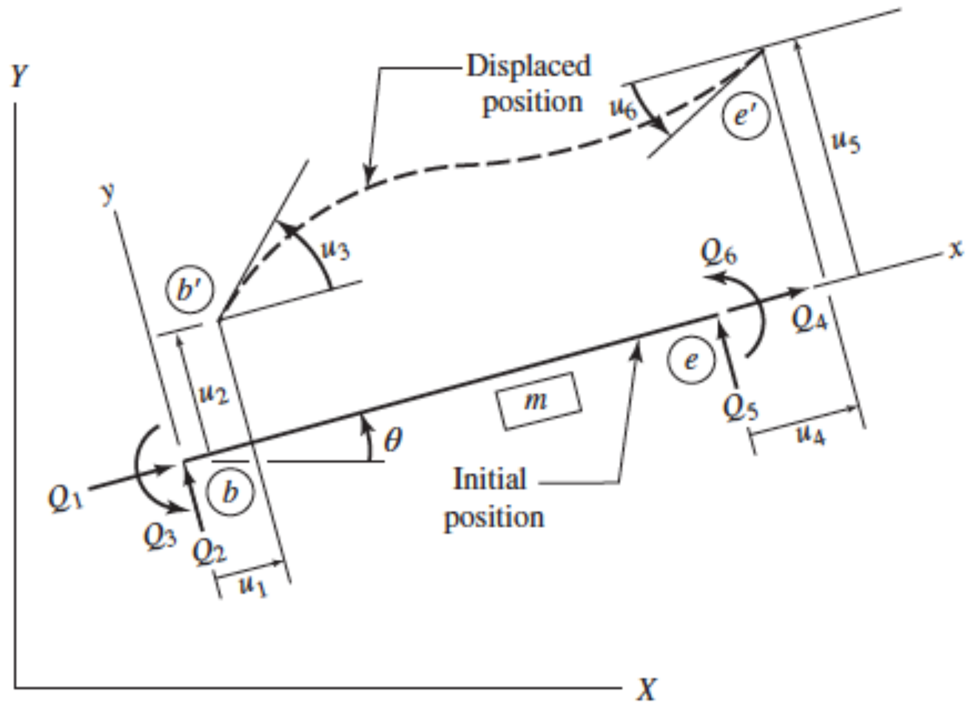
$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{21} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} + \begin{Bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \\ Q_{f5} \\ Q_{f6} \end{Bmatrix}$$

Coordinate transformations (3)

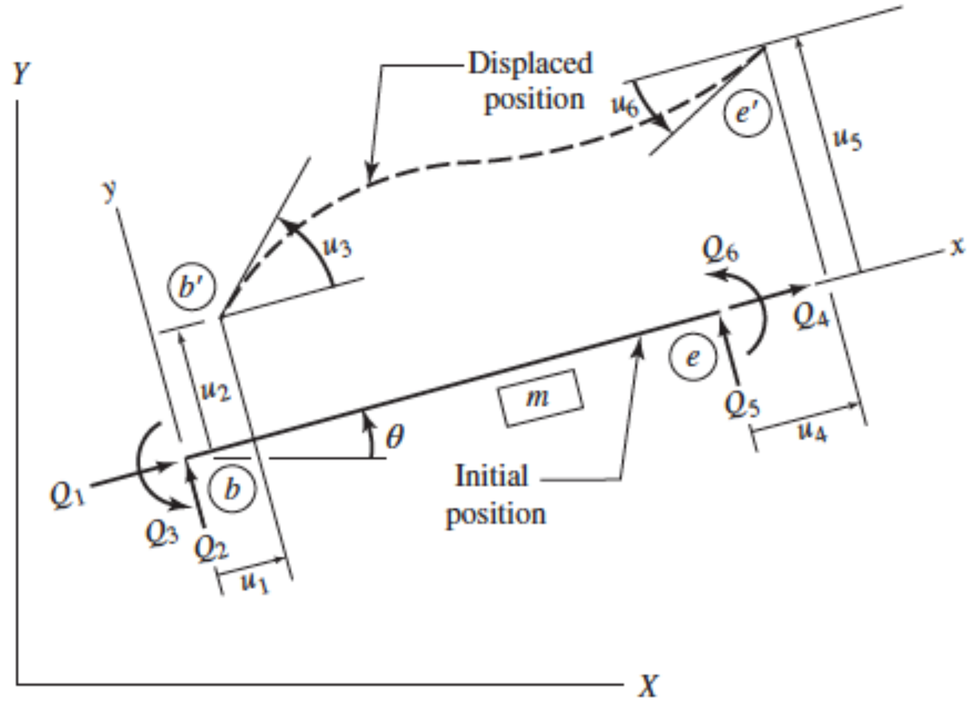
$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}$$

Coordinate transformations (4)





Coordinate transformations (5)



$$Q_1 = F_1 cos\theta + F_2 sin\theta$$

$$Q_2 = -F_1 sin\theta + F_2 cos\theta$$

$$Q_3 = F_3$$

$$Q_4 = F_4 cos\theta + F_5 sin\theta$$

$$Q_5 = -F_4 sin\theta + F_5 cos\theta$$

$$Q_6 = F_6$$

Coordinate transformations (6)

Force transformation

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \underbrace{\begin{bmatrix} cos\theta & sin\theta & 0 & 0 & 0 & 0 \\ -sin\theta & cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & cos\theta & sin\theta & 0 \\ 0 & 0 & 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix}$$

$\mathbf{T}$ Transformation matrix

$$\boldsymbol{Q} = \mathbf{T}\ \mathbf{F}$$

$$cos\theta = \frac{x_e - x_b}{L} = \frac{x_e - x_b}{\sqrt{(x_e - x_b)^2 + (y_e - y_b)^2}} \qquad sin\theta = \frac{y_e - y_b}{L} = \frac{y_e - y_b}{\sqrt{(x_e - x_b)^2 + (y_e - y_b)^2}}$$

Coordinate transformations (3)

End displacement transformation

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \underbrace{\begin{bmatrix} cos\theta & sin\theta & 0 & 0 & 0 & 0 \\ -sin\theta & cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & cos\theta & sin\theta & 0 \\ 0 & 0 & 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}\ \text{Transformation matrix}} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix}$$

$$\mathbf{\Delta} = \mathbf{T}\,\boldsymbol{\nu}$$

Transformation from local to global coordinate systems

The transformation matrix $\mathbf{T}$ is an orthogonal matrix; thus,

$$\mathbf{T}^{-1} = \mathbf{T}^{T}$$

Forces from local to global coordinate system:

$$\mathbf{F} = \mathbf{T}^{-1}\boldsymbol{Q} = \mathbf{T}^{T}\boldsymbol{Q}$$

Displacements from local to global coordinate system:

$$\boldsymbol{v} = \mathbf{T}^{T}\,\boldsymbol{\Delta}$$

Member stiffness relations in the global coordinate system

Forces from local to global coordinate system:

$$\mathbf{F} = \mathbf{T}^T \boldsymbol{Q} = \mathbf{T}^T \mathbf{k} \boldsymbol{\Delta} + \mathbf{T}^T \mathbf{Q}_f$$

Local stiffness matrix

$$\boldsymbol{\Delta} = (\mathbf{T}^T \mathbf{k})^{-1} (\mathbf{F} - \boldsymbol{T}^T \mathbf{Q}_f) = \mathbf{k}^{-1} (\mathbf{T}^T)^{-1} (\mathbf{F} - \boldsymbol{T}^T \mathbf{Q}_f) = \mathbf{k}^{-1} \mathbf{T} (\mathbf{F} - \boldsymbol{T}^T \mathbf{Q}_f)$$

Displacements from local to global coordinate system:

$$\boldsymbol{\nu} = \mathbf{T}^T \boldsymbol{\Delta} = \mathbf{T}^T \mathbf{k}^{-1} \mathbf{T} (\mathbf{F} - \boldsymbol{T}^T \mathbf{Q}_f) = (\mathbf{T}^T \mathbf{k}^{-1} \mathbf{T})(\mathbf{F} - \boldsymbol{T}^T \mathbf{Q}_f)$$

Member flexibility matrix in the global coordinate system

$$\mathbf{K}^{-1} = (\mathbf{T}^T \mathbf{k}^{-1} \mathbf{T})$$

Member stiffness matrix

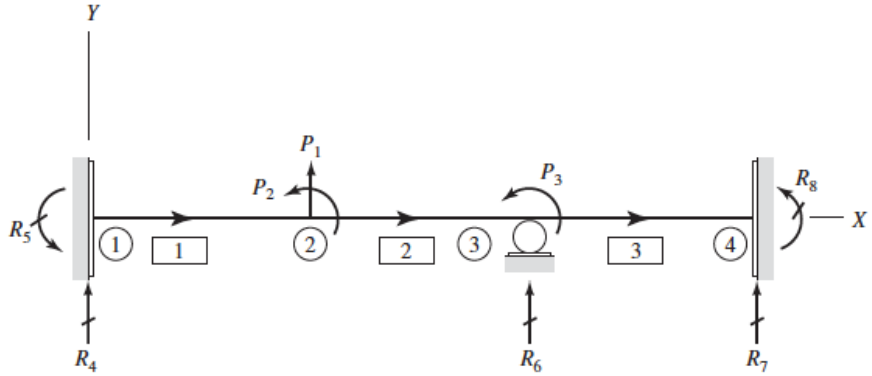
$$\mathbf{K} = (\mathbf{T}^T \mathbf{k} \mathbf{T})$$

Member stiffness relations in the global coordinate system

$$
\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix}
\frac{AL^2}{I}\cos^2\theta + 12\sin^2\theta & \left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & -6L\sin\theta & -\left(\frac{AL^2}{I}\cos^2\theta + 12\sin^2\theta\right) & -\left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & -6L\sin\theta \\
\left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & \frac{AL^2}{I}\sin^2\theta + 12\cos^2\theta & 6L\cos\theta & -\left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & -\left(\frac{AL^2}{I}\sin^2\theta + 12\cos^2\theta\right) & 6L\cos\theta \\
-6L\sin\theta & 6L\cos\theta & 4L^2 & 6L\sin\theta & -6L\cos\theta & 2L^2 \\
-\left(\frac{AL^2}{I}\cos^2\theta + 12\sin^2\theta\right) & -\left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & 6L\sin\theta & \frac{AL^2}{I}\cos^2\theta + 12\sin^2\theta & \left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & 6L\sin\theta \\
-\left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & -\left(\frac{AL^2}{I}\sin^2\theta + 12\cos^2\theta\right) & -6L\cos\theta & \left(\frac{AL^2}{I} - 12\right)\cos\theta\sin\theta & \frac{AL^2}{I}\sin^2\theta + 12\cos^2\theta & -6L\cos\theta \\
-6L\sin\theta & 6L\cos\theta & 2L^2 & 6L\sin\theta & -6L\cos\theta & 4L^2
\end{bmatrix}
$$

Assembly of the structure stiffness matrix

-Code number technique applies



Three degrees of freedom (1 through 3);
five restrained coordinates (4 through 8)

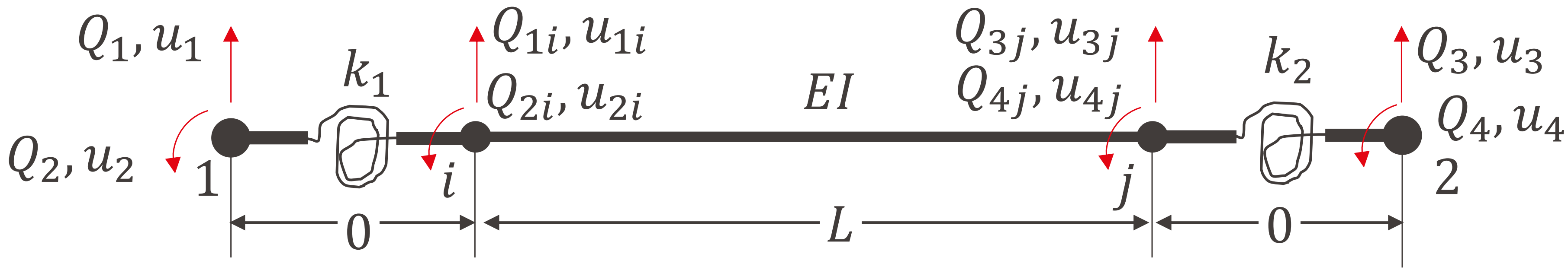
(a) Analytical Model

$$\mathbf{k}_1 = \begin{array}{c} \begin{array}{cccc} 4 & 5 & 1 & 2 \end{array} \\ \left[\begin{array}{cccc} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} \\ k_{31}^{(1)} & k_{32}^{(1)} & k_{33}^{(1)} & k_{34}^{(1)} \\ k_{41}^{(1)} & k_{42}^{(1)} & k_{43}^{(1)} & k_{44}^{(1)} \end{array}\right] \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \end{array} \qquad \mathbf{k}_2 = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 6 & 3 \end{array} \\ \left[\begin{array}{cccc} k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} & k_{14}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & k_{23}^{(2)} & k_{24}^{(2)} \\ k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} & k_{34}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)} & k_{44}^{(2)} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 6 \\ 3 \end{array} \qquad \mathbf{k}_3 = \begin{array}{c} \begin{array}{cccc} 6 & 3 & 7 & 8 \end{array} \\ \left[\begin{array}{cccc} k_{11}^{(3)} & k_{12}^{(3)} & k_{13}^{(3)} & k_{14}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} & k_{23}^{(3)} & k_{24}^{(3)} \\ k_{31}^{(3)} & k_{32}^{(3)} & k_{33}^{(3)} & k_{34}^{(3)} \\ k_{41}^{(3)} & k_{42}^{(3)} & k_{43}^{(3)} & k_{44}^{(3)} \end{array}\right] \end{array} \begin{array}{c} \\ 6 \\ 3 \\ 7 \\ 8 \end{array}$$

$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} k_{33}^{(1)} + k_{11}^{(2)} & k_{34}^{(1)} + k_{12}^{(2)} & k_{14}^{(2)} \\ k_{43}^{(1)} + k_{21}^{(2)} & k_{44}^{(1)} + k_{22}^{(2)} & k_{24}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{44}^{(2)} + k_{22}^{(3)} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

(b) Assembling of Structure Stiffness Matrix S

Elastic beam element with rotational springs



Linear rotational spring to consider the connection flexibility within a numerical model

$$M = k\theta_r$$

Left end

$$Q_2 = k_1(u_2 - u_{2i})$$

Right end

$$Q_4 = k_2(u_4 - u_{4j})$$

The length of the connection is ignored, and the spring stiffness *k* is generally taken as an empirically (or semi-empirically) determined initial stiffness of the connection (usually based on experimental evidence).

Elastic beam-column element with rotational springs

Let's assume the following:

$$k_1 = a_1 EI/L \qquad k_2 = a_2 EI/L \qquad a = \frac{a_1 a_2}{a_1 a_2 + 4a_1 + 4a_2 + 12}$$

The "spring and beam" element stiffness matrix becomes:

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \frac{aEI}{L} \begin{bmatrix} \frac{12}{L^2}\left(1 + \frac{a_1 + a_2}{a_1 a_2}\right) & \frac{6}{L}\left(1 + \frac{2}{a_2}\right) & -\frac{12}{L^2}\left(1 + \frac{a_1 + a_2}{a_1 a_2}\right) & \frac{6}{L}\left(1 + \frac{2}{a_1}\right) \\ \frac{6}{L}\left(1 + \frac{2}{a_2}\right) & 4\left(1 + \frac{3}{a_2}\right) & -\frac{6}{L}\left(1 + \frac{2}{a_2}\right) & 2 \\ -\frac{12}{L^2}\left(1 + \frac{a_1 + a_2}{a_1 a_2}\right) & -\frac{6}{L}\left(1 + \frac{2}{a_2}\right) & \frac{12}{L^2}\left(1 + \frac{a_1 + a_2}{a_1 a_2}\right) & -\frac{6}{L}\left(1 + \frac{2}{a_1}\right) \\ \frac{6}{L}\left(1 + \frac{2}{a_1}\right) & 2 & -\frac{6}{L}\left(1 + \frac{2}{a_1}\right) & 4\left(1 + \frac{3}{a_1}\right) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Two typical cases

Fixed-end beam $a_1 \sim \infty$ and $a_2 \sim \infty$

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} \frac{12}{L^2} & \frac{6}{L} & -\frac{12}{L^2} & \frac{6}{L} \\ \frac{6}{L} & 4 & -\frac{6}{L} & 2 \\ -\frac{12}{L^2} & -\frac{6}{L} & \frac{12}{L^2} & -\frac{6}{L} \\ \frac{6}{L} & 2 & -\frac{6}{L} & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Fix-pinned beam $a_1 \sim \infty$ and $a_2 \sim 0$

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \frac{EI}{L} \begin{bmatrix} 3/L^2 & 3/L^2 & -3/L^2 & 0 \\ 3/L & 3 & -3/L & 0 \\ -3/L^2 & -3/L & 3/L^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

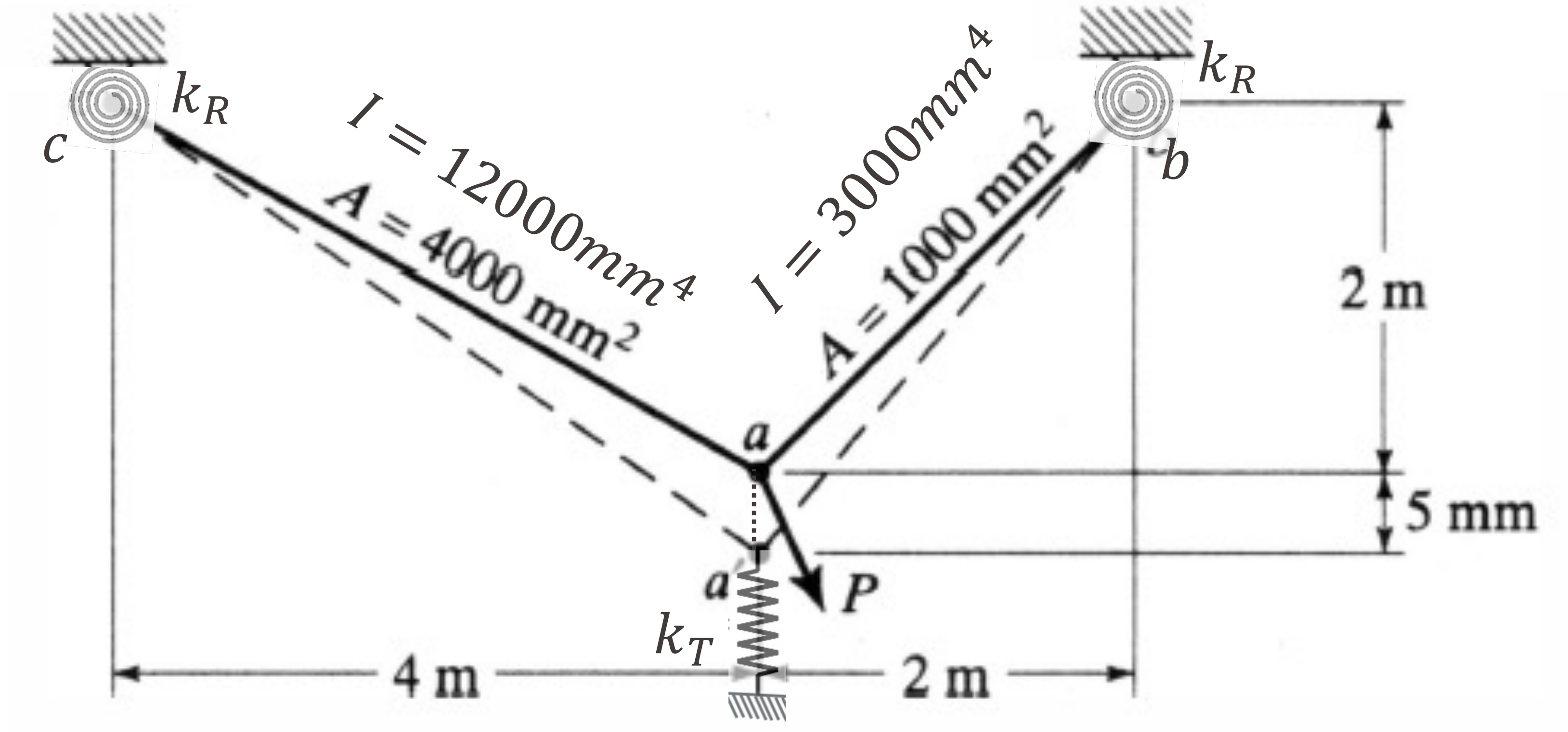
Some remarks about "infinity"

- Usually, you can assume something is "infinitely still" if it is 1 to 2 orders of magnitude larger than the elastic beam element flexural stiffness; therefore,

$$k_1 \sim (10\ to 100) \cdot EI/L$$

- If you plan to use a zero-length element, software does not like "infinitely" stiff elements due to numerical convergence.

- Commercial software assume the above value but inherently imply that the zero-length elastic stiffness is theoretically infinite (Not true).



k_R
c
$I = 12000mm^4$
$A = 4000$ mm^2
$I = 3000mm^4$
$A = 1000$ mm^2
k_R
b
2 m
a
5 mm
a
P
k_T
4 m
2 m