# CIVIL 449: Nonlinear Analysis of Structures

School of Architecture, Civil & Environmental Engineering
Civil Engineering Institute

## Geometric Nonlinear Analysis

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# Objectives of today's lecture

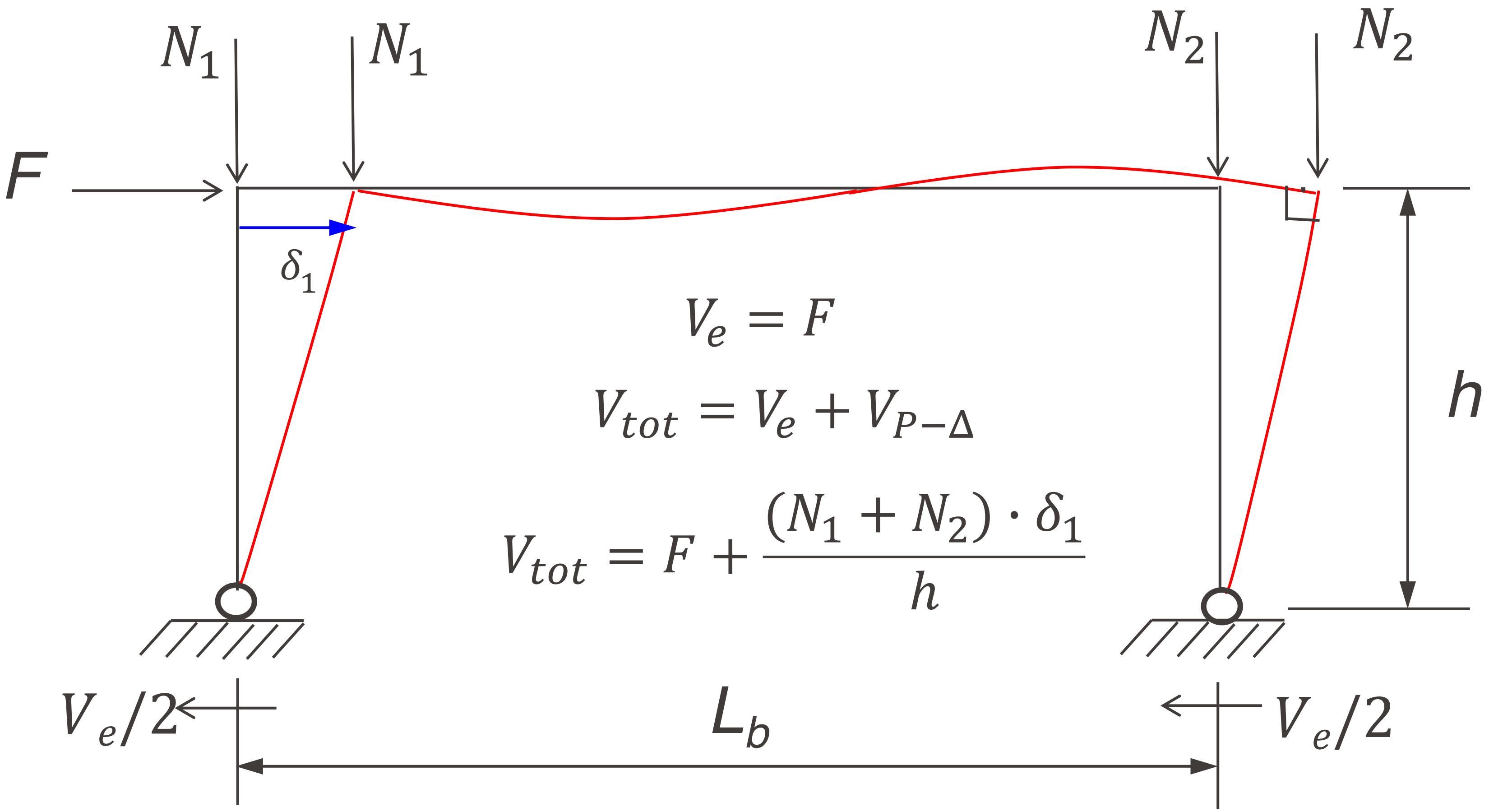
- To introduce:
  - Geometric stiffness matrix
  - Basic reference system for frame elements
  - Element transformations (from local to basic coordinate system)
  - Corotational transformation

# Deflections

- Strength and stiffness are completely different properties of a member, that are indeed related to each other.
- A fishing rod is flexible yet strong.
- Floor systems and structures cannot deflect as much for several reasons.

*Source: http://www.africancichlid.com/Structure.htm*

# Deflections cause second order effects

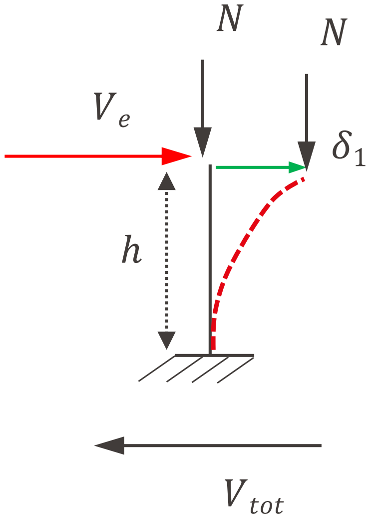


$$V_e = F$$

$$V_{tot} = V_e + V_{P-\Delta}$$

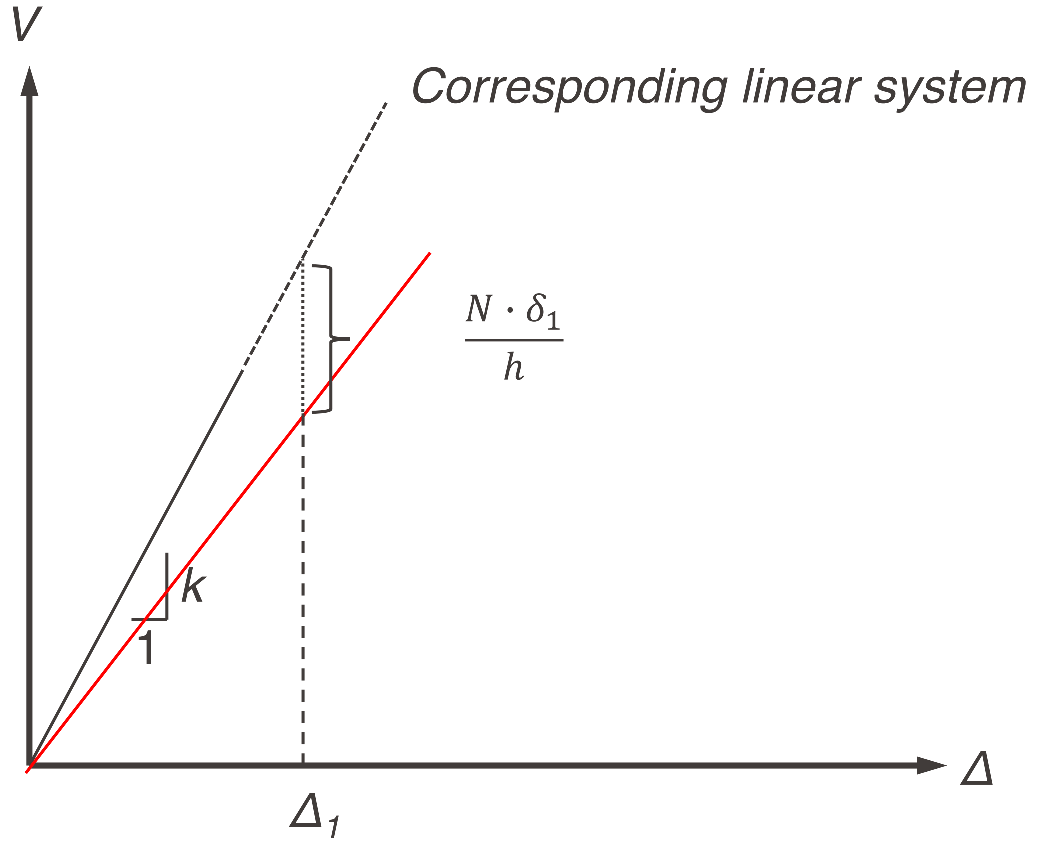
$$V_{tot} = F + \frac{(N_1 + N_2) \cdot \delta_1}{h}$$

# Deflections cause second order effects (2)

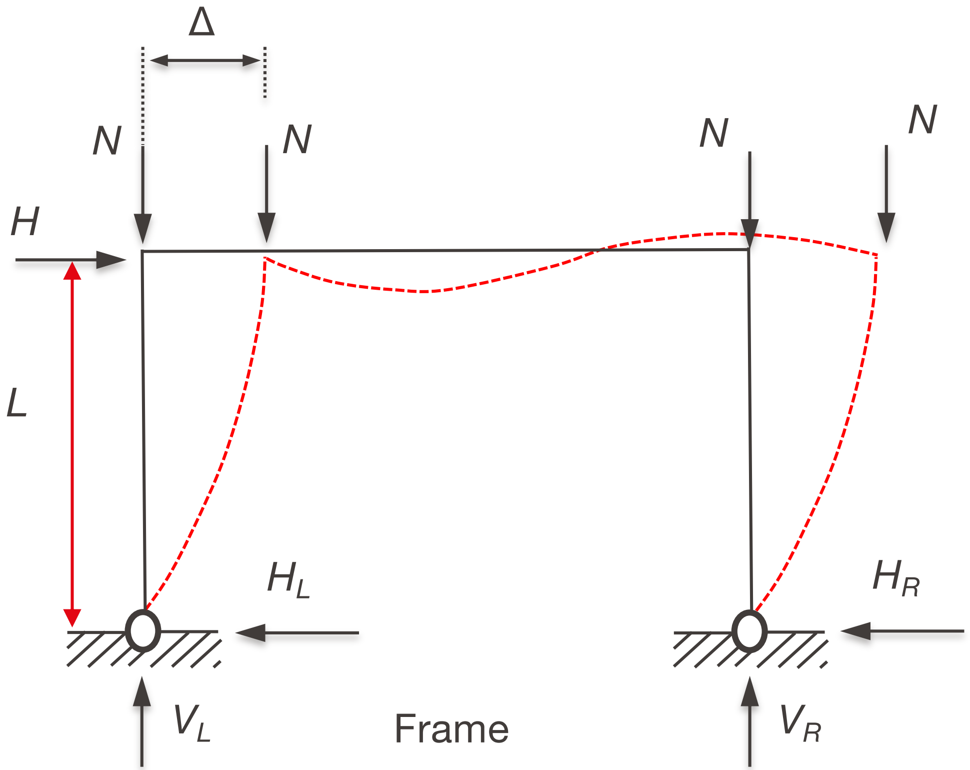


$$V_{tot} = V_e + V_{P-\Delta}$$

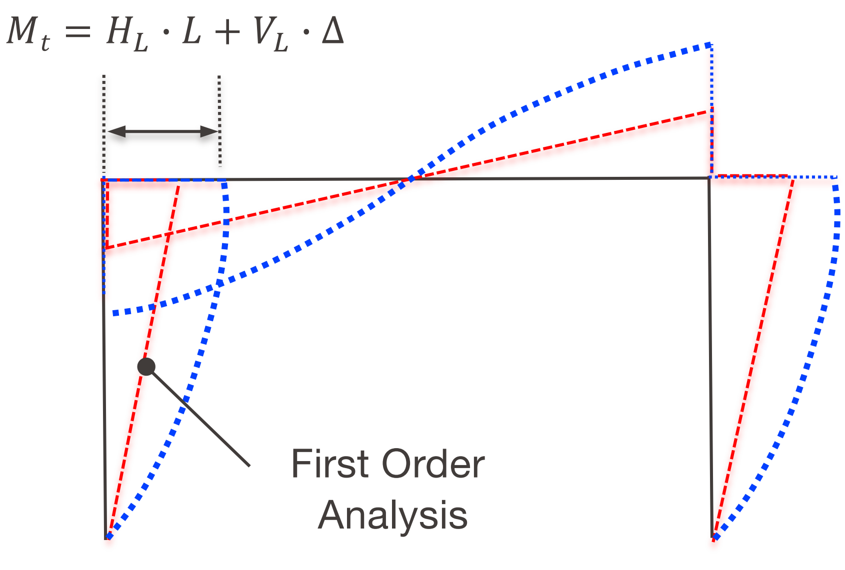
$$V_{tot} = V_e + \frac{N \cdot \delta_1}{h}$$



# P-Delta effects on frame structures



Frame



$$M_t = H_L \cdot L + V_L \cdot \Delta$$

Bending Moments

- $N \cdot \Delta$: Additional moment (couple) due to the axial force acting through the relative transverse displacement of member ends

# Types of analysis

1. **First Order Elastic Analysis**: The equations of equilibrium are always written in the undeformed configuration and material nonlinearity is not considered.

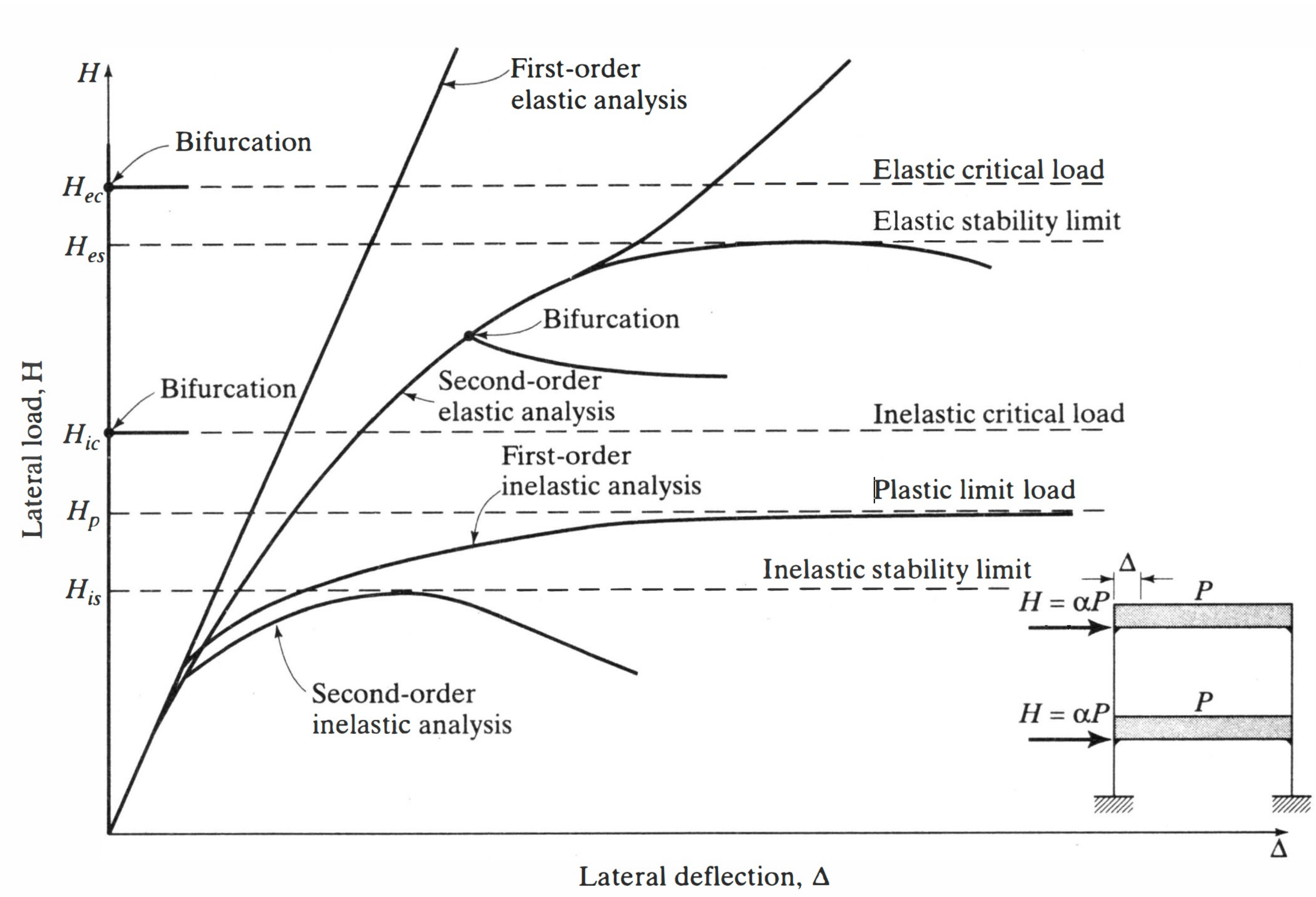
Normally they require special software

2. **Second Order Elastic Analysis**: The equations of equilibrium are always written in the deformed configuration and material nonlinear is not considered.

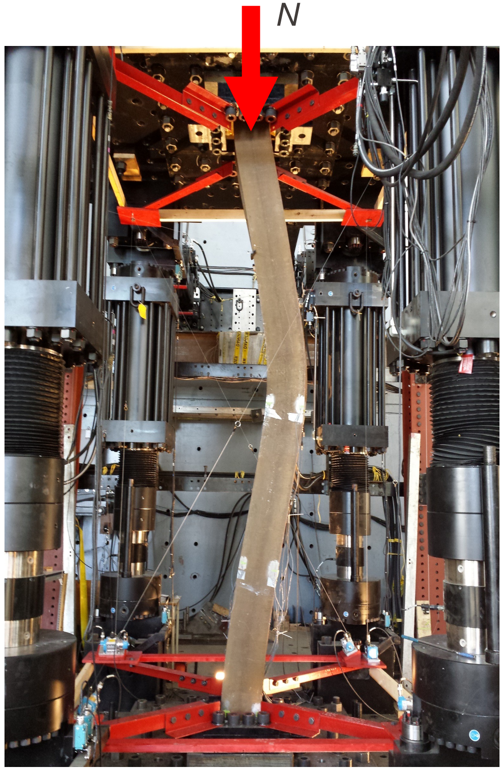
3. **First Order inelastic Analysis**: The equations of equilibrium are always written in the undeformed configuration and material nonlinearity is considered.

4. **Second Order Inelastic Analysis**: The equations of equilibrium are always written in the deformed configuration and material nonlinearity is considered.

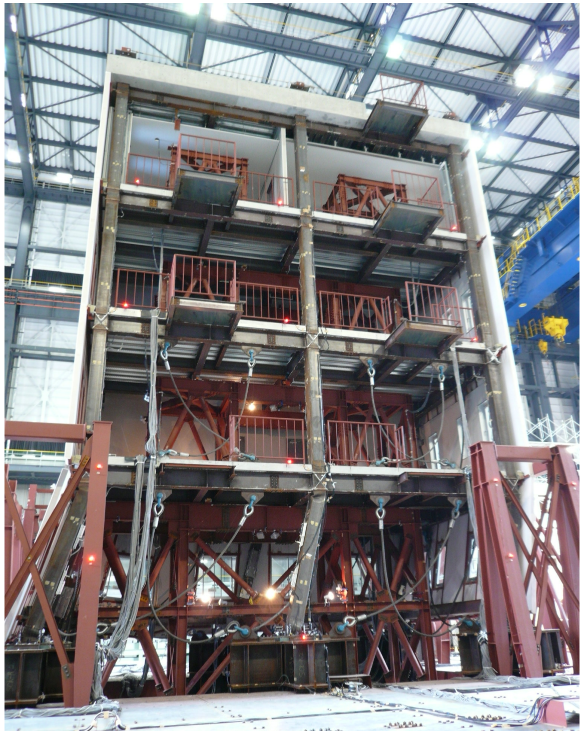
# Types of analysis (2)



# Several examples from established theory



@Prof. R. Tremblay (2015)

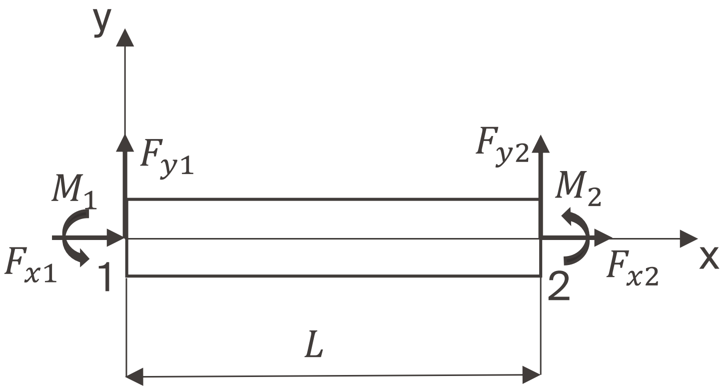


Lignos et al. (2013)

@Prof. E. Miranda (2017)

# Geometric stiffness matrix for planar elements

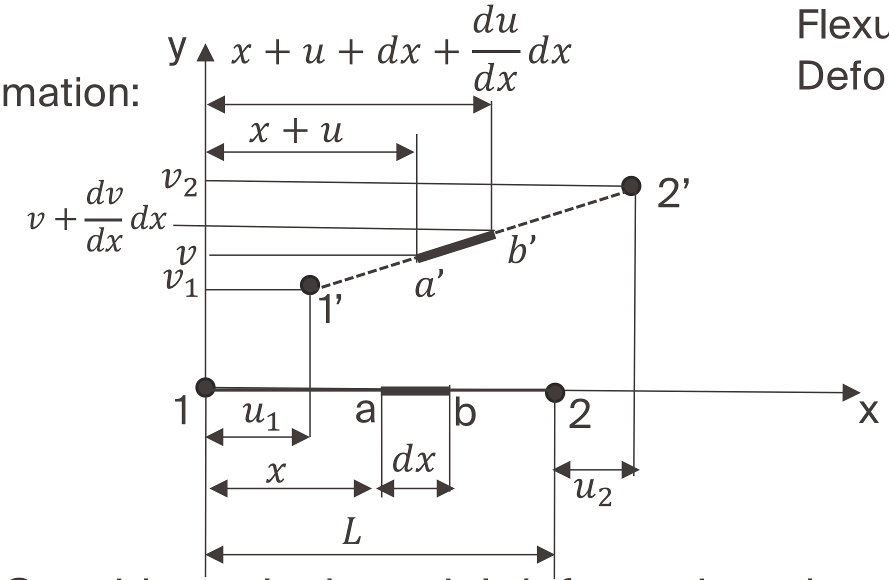
- Instead of using infinitesimal strains that form the basis of linear analysis, start using small finite strains
- Combination of bending and axial force
- Consider the 2d elastic beam element that we saw previously:



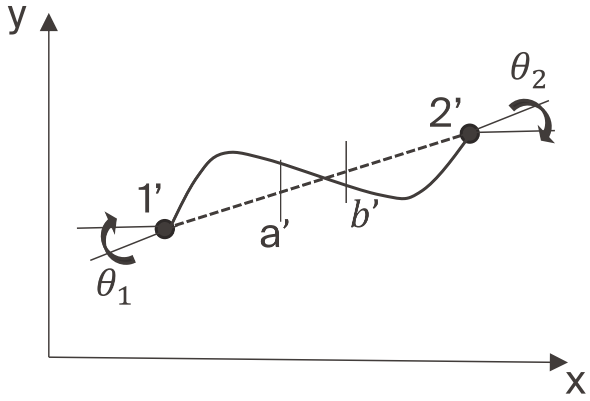
# Geometric stiffness matrix for planar elements (2)

- Consider the effect of both axial and flexural deformations

Axial Deformation:



Flexural Deformation:



- Consider only the axial deformation; denote by $ab$ the length of the infinitesimal segment $dx$ (i.e., $ab = dx$ ) in the reference configuration
- After rigid body rotation and axial deformation, the length of the segment $dx$ is as follows:

$$a'b' = \left[\left(dx + \frac{du}{dx}dx\right)^2 + \left(\frac{dv}{dx}dx\right)^2\right]^{\frac{1}{2}} = \left[1 + 2\frac{du}{dx} + \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right]^{\frac{1}{2}} dx$$

# Geometric stiffness matrix for planar elements (3)

- Using the binomial theorem and neglecting the higher-order terms gives

$$\frac{a'b'}{dx} = 1 + \frac{du}{dx} + \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right]$$

- The finite strain $e_{fin}$ is defined as the sum of the extension per unit length (for the axial deformation) and the strain form the flexural deformation:

$$e_{fin} = \frac{a'b' - ab}{dx} - y\left(\frac{d^2v}{dx^2}\right) = \frac{du}{dx} + \frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right] - y\left(\frac{d^2v}{dx^2}\right)$$

Where the last term ($\frac{d^2v}{dx^2}$:= curvature) is the infinitesimal flexural strain (i.e., neglect the effects of the rotation and stretching of the element neutral axis)

- The theorem of virtual displacements (for a uniaxial stress state) is applied to the reference configuration:

$$\delta W_{int} = \int_V \sigma_x \delta e_{fin} dV$$

- The following relations are useful for the next step:

$$\sigma_x = -\frac{yM}{I} \text{ with } I = \int_A y^2 dA$$

$\delta\left(\frac{du}{dx}\right) = \frac{d\delta u}{dx}$ and $\delta\left(\frac{dv}{dx}\right) = \frac{d\delta v}{dx}$ which are valid for infinitesimal displacements

# Geometric stiffness matrix for planar elements (4)

- Using the definition of $e_{fin}$, the previous quantities and integrating over the section depth gives

$$\delta W_{int} = \int_0^L \sigma_x A \left(\frac{d\delta u}{dx}\right) dx + \int_0^L M \left(\frac{d^2 v}{dx^2}\right) dx + \frac{1}{2}\int_0^L \sigma_x A \left[\delta\left(\frac{du}{dx}\right)^2 + \delta\left(\frac{dv}{dx}\right)^2\right] dx$$

- The following relations are useful for the next step:

$$\sigma_x = \left(\frac{du}{dx}\right) E,\ F_{x2} = \sigma_x A,\ M = \left(\frac{d^2 v}{dx^2}\right) EI$$

- The virtual work can be rewritten as

$$\delta W_{int} = \int_0^L \left(\frac{du}{dx}\right) EA \left(\frac{d\delta u}{dx}\right) dx + \int_0^L \left(\frac{d^2 v}{dx^2}\right) EI \left(\frac{d^2 v}{dx^2}\right) dx + \frac{1}{2} F_{x2} \int_0^L \left[\delta\left(\frac{du}{dx}\right)^2 + \delta\left(\frac{dv}{dx}\right)^2\right] dx$$

- The elastic stiffness matrix $\mathbf{k}_e$ (both for axial and flexural deformation) follows from the first two integrals
- The third integral produces a geometric stiffness matrix $\mathbf{k}_g$
- To compute the third integral, the "mathematical trick" is that the virtual operator $\delta$ may be threated as a differential operator with respect to the variables $\frac{du}{dx}$ and $\frac{dv}{dx}$ → for variable $u$: $\delta\left(\frac{du}{dx}\right)^2 = 2\,\frac{d\delta u}{dx}\,\frac{du}{dx}$

# Geometric stiffness matrix for planar elements (5)

- The third term of the virtual work can then be rewritten as

$$\delta W_{int,3} = F_{x2} \int_0^L \left[ \left( \frac{d\delta u}{dx} \frac{du}{dx} \right) + \left( \frac{d\delta v}{dx} \frac{dv}{dx} \right) \right] dx$$

- The displacements are interpolated using the usual shape functions $u\left(\frac{x}{L}\right) = \mathbf{Nu}$ :
  - For an axial member:

$$\mathbf{u} = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

  - For a flexural  member (see lecture notes from last week):

$$v = \begin{bmatrix} 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 & x - 2x\left(\frac{x}{L}\right) + x\left(\frac{x}{L}\right)^2 & 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 & \frac{x^2}{L}\left(-1 + \frac{x}{L}\right) \end{bmatrix} \begin{pmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{pmatrix}$$

- The derivative of the displacement with respect to $x$ are given by

$$\frac{du}{dx} = \frac{d\mathbf{N}}{dx}\mathbf{u} = \mathbf{N}'\mathbf{u}$$

- Similarly,

$$\delta u = \mathbf{N}\delta\mathbf{u} \text{ and } \frac{d\delta u}{dx} = \mathbf{N}'\delta\mathbf{u}$$

# Geometric stiffness matrix for planar elements (6)

- The third term of the virtual work can then be rewritten as

$$\delta W_{int,3} = \delta \mathbf{u}^T \left( F_{x2} \int_0^L [\mathbf{N'}_u^T \mathbf{N'}_u + \mathbf{N'}_v^T \mathbf{N'}_v] dx \right) \mathbf{u}$$

Where $\mathbf{N}_u$ and $\mathbf{N}_\mathrm{v}$ are the row vectors of the shape functions for the axial and flexural member, respectively.

NOTE: The (+) sign shall be interpreted as an assembly procedure for the corresponding degrees of freedom.

- Recognizing that from the virtual work theorem, the internal work should be equal to the external work $\delta W_{ext} = \delta \mathbf{u}^T \mathbf{F}$ and using $\mathbf{F} = \mathbf{ku}$, the following local geometric stiffness matrix is obtained:

$$\mathbf{k}_\mathrm{g}^{local} = F_{x2} \int_0^L [\mathbf{N'}_u^T \mathbf{N'}_u + \mathbf{N'}_v^T \mathbf{N'}_v] dx$$

# Geometric stiffness matrix for planar elements (7)

- And after multiplying and integrating

$$\mathbf{k}_{\mathrm{g}}^{local} = \frac{F_{x2}}{L} \begin{array}{cc} \begin{array}{cccccc} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{L}{10} & 0 & -\frac{6}{5} & \frac{L}{10} \\ 0 & \frac{L}{10} & \frac{2L^2}{15} & 0 & -\frac{L}{10} & -\frac{L^2}{30} \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{6}{5} & -\frac{L}{10} & 0 & \frac{6}{5} & -\frac{L}{10} \\ 0 & \frac{L}{10} & -\frac{L^2}{30} & 0 & -\frac{L}{10} & \frac{2L^2}{15} \end{bmatrix} & \begin{array}{c} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{array} \end{array}$$

- To assemble the local geometric stiffness matrices to the global geometric stiffness matrix, the usual assembly procedure presented in the previous lectures may be used.

# Elastic stiffness matrix for planar elements

- Recall, the elastic stiffness matrix $\mathbf{k}_e^{local}$ for elastic 2d beam elements (presented in the previous lecture) is given as a reminder:

$$\mathbf{k}_e^{local} = \frac{EI}{L^3}\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccc|c} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 & \\ \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 & u_1 \\ 0 & 12 & 6L & 0 & -12 & 6L & v_1 \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 & \theta_1 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 & u_2 \\ 0 & -12 & -6L & 0 & 12 & -6L & v_2 \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 & \theta_2 \end{array}$$

# Geometric stiffness matrix

- The geometric stiffness matrix can be used to compute the elastic critical loads which will lead to flexural buckling and/or lateral torsional buckling. The predicted instability mode depends on the degrees of freedom present in the element formulation.

- To compute the elastic critical load, the global stiffness equation is written in the form of a generalized eigenvalue problem; the equation of equilibrium at the critical state is:

$$\left[\mathbf{K}_e^{global} + \lambda \widehat{\mathbf{K}}_g^{global}\right]\boldsymbol{\Delta} = \mathbf{0}$$

where $\mathbf{K}_e^{global}$ is the global elastic stiffness matrix, $\widehat{\mathbf{K}}_g^{global}$ is the global geometric stiffness matrix computed for a reference load, $\mathbf{P}_{ref}$; $\lambda$ (an eigenvalue) is the load factor with respect to $\mathbf{P}_{ref}$ and $\boldsymbol{\Delta}$ (an eigenvector) is the buckled shape.

- The lowest value of $\lambda$ that satisfies the equation for $\boldsymbol{\Delta} \neq \mathbf{0}$ gives the elastic critical load vector $\lambda \boldsymbol{P}_{ref}$ and the corresponding, $\boldsymbol{\Delta}$ defines the buckled configuration.

# Solution of the eigenvalue problem

- To determine the eigenvalues and eigenvector, the eigenvalue problem is rewritten as

$$\mathbf{K}_{ef}^{global}\boldsymbol{\Delta}_{f} = -\lambda\widehat{\mathbf{K}}_{gf}^{global}\boldsymbol{\Delta}_{f} \quad \Leftrightarrow \frac{1}{\lambda}\mathbf{K}_{ef}^{global}\boldsymbol{\Delta}_{f} = -\widehat{\mathbf{K}}_{gf}^{global}\boldsymbol{\Delta}_{f} \quad \Leftrightarrow -\left(\mathbf{K}_{ef}^{global}\right)^{-1}\widehat{\mathbf{K}}_{gf}^{global}\boldsymbol{\Delta}_{f} = \frac{1}{\lambda}\boldsymbol{\Delta}_{f}$$

where the subscript $f$ indicates that the respective matrices and vectors relate to the free degrees of freedom only.

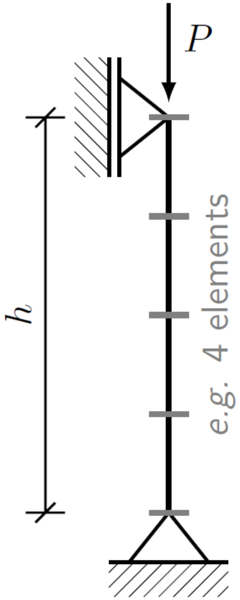
# Solution of the eigenvalue problem (2)

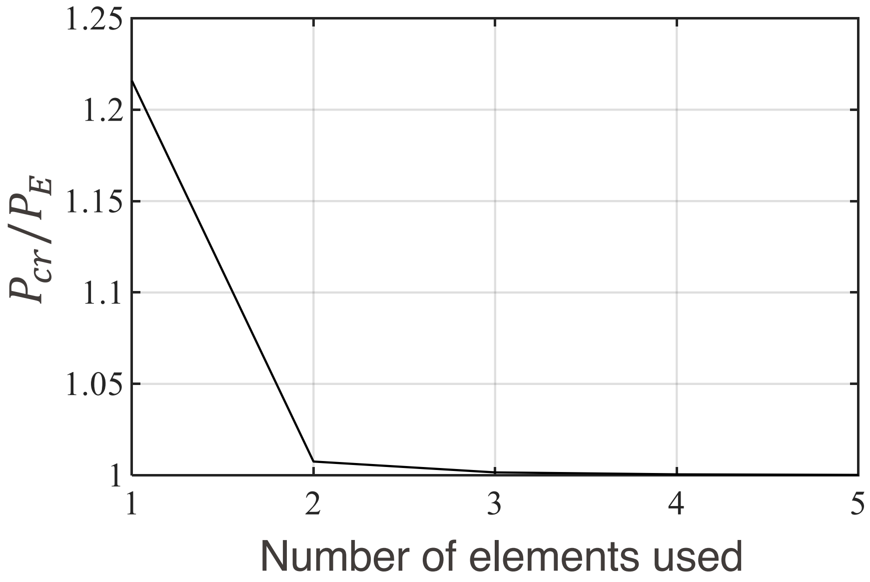
- The following approach may be used to determine the elastic critical loads of a structure:
  1. Assemble the global elastic stiffness matrix of the structure $\mathbf{k}_e^{global}$
  2. Impose the boundary conditions (external unit loads $\mathbf{F}$)
  3. Using the static condensation, solve the system $\mathbf{F} = \mathbf{K}_e^{global}\mathbf{v}$ to determine the nodal displacements $\mathbf{v}$
  4. For each element:
     a. In the local reference system, solve the system $\mathbf{Q} = \mathbf{k}_e^{local}\mathbf{u}$ to determine the internal forces $\mathbf{Q}$
     b. Compute the local geometric stiffness matrix $\mathbf{k}_g^{local}$
  5. Assemble the global geometric stiffness matrix of the structure $\mathbf{k}_g^{global}$
  6. Solve the eigenvalue problem $\left(\mathbf{k}_{ef}^{global}\right)^{-1}\mathbf{k}_g^{global}\boldsymbol{\Delta}_{\boldsymbol{f}} = \frac{1}{\lambda}\boldsymbol{\Delta}_{\boldsymbol{f}}$ to determine the load multipliers $\lambda$
  7. The critical load $\mathbf{P}_{cr}$ is obtained by taking the minimum (in absolute value) load multiplier $\lambda_{\min} = \min(|\lambda|)$ and multiplying it with the applied unit load $\mathbf{F}$:

$$\mathbf{P}_{cr} = \lambda_{\min}\mathbf{F}$$

# Example: Euler buckling load

- Determine the Euler load of a $h = 3.0m$ high Euler column. Assume a HEA 320 steel cross section ($\mathrm{E} = 200\ \mathrm{G}Pa, I = 229.3 \cdot 10^6\ mm^4$ and $A = 12400\ mm^2$)
- Use 2d elastic beam elements (see previous slides for elastic and geometric stiffness matrices)
- Compare with the theoretical Euler buckling load $P_E$ given by, $P_E = \frac{\pi^2 EI}{h^2}$



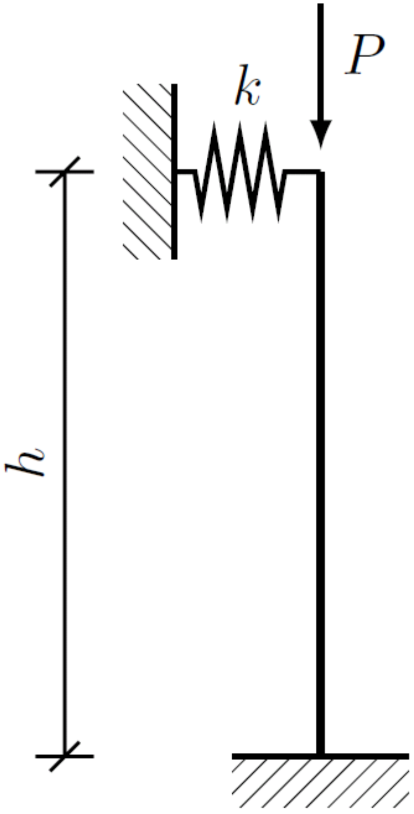


# Example: Effect of the lateral restraint on the buckling load

- What is the influence of the spring stiffness on the buckling response of the column?
- The buckling load $P_{cr}$ is given by

$$P_{cr} = \frac{\pi^2 EI}{(Kh)^2}$$

Where $k$ is the effective length factor ($Kh$ is the buckling length of the column)



- The stiffness $k$ of the spring is taken as

$$k = \alpha \frac{3EI}{h^3}$$

Where $3EI/h^3$ is the lateral (translational) stiffness of the cantilever column

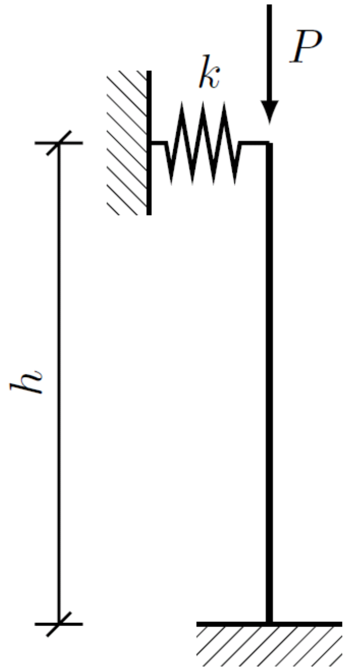
- Include the effect of the spring when assembling the global elastic stiffness matrix

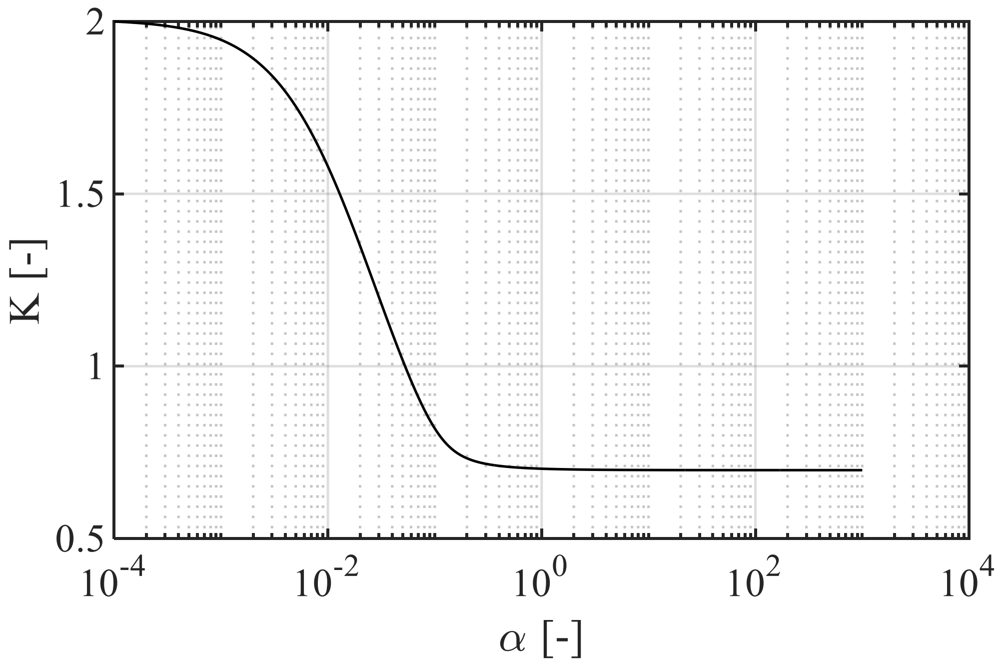
- To determine the influence of the spring stiffness on the buckling length when computing the buckling load, compute the effective length factor $K$ using

$$K = \sqrt{\frac{\pi^2 EI}{P_{cr} h^2}}$$

Where $P_{cr}$ is obtained by solving the eigenvalue problem

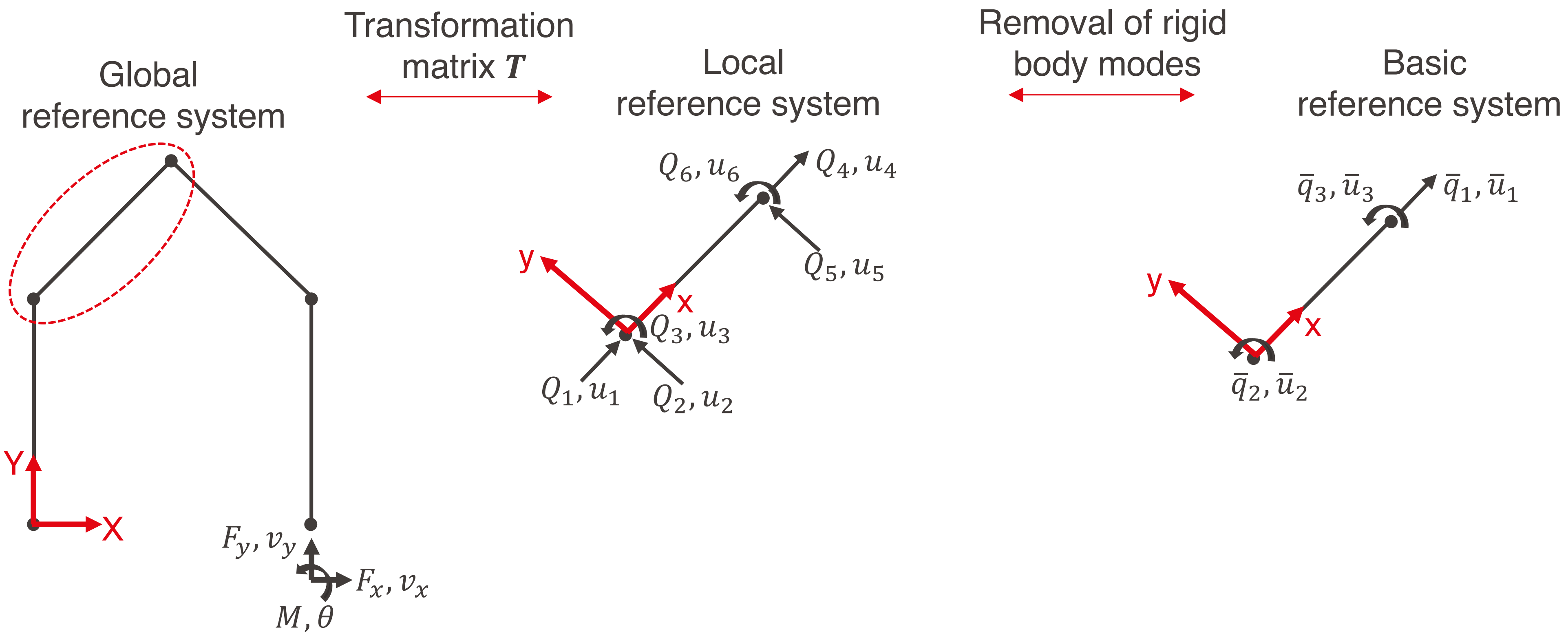
# Example: Effect of the lateral restraint on the buckling load (2)





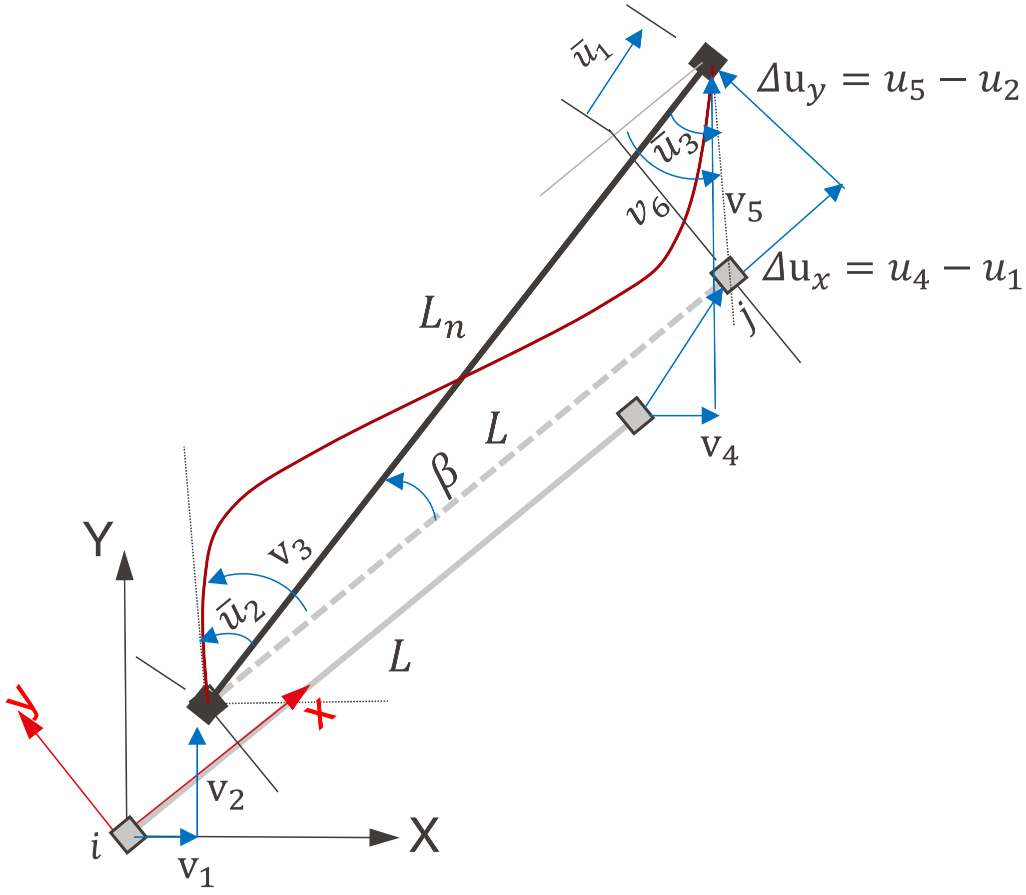
- The effective length factor $K$ varies from $K = 2$ to $K = 0{,}7$
- The value $K = 2$ corresponds to a fixed-free column
- The value $K = 0{,}7$ corresponds to a fixed-fixed column

# Basic reference system for frame elements

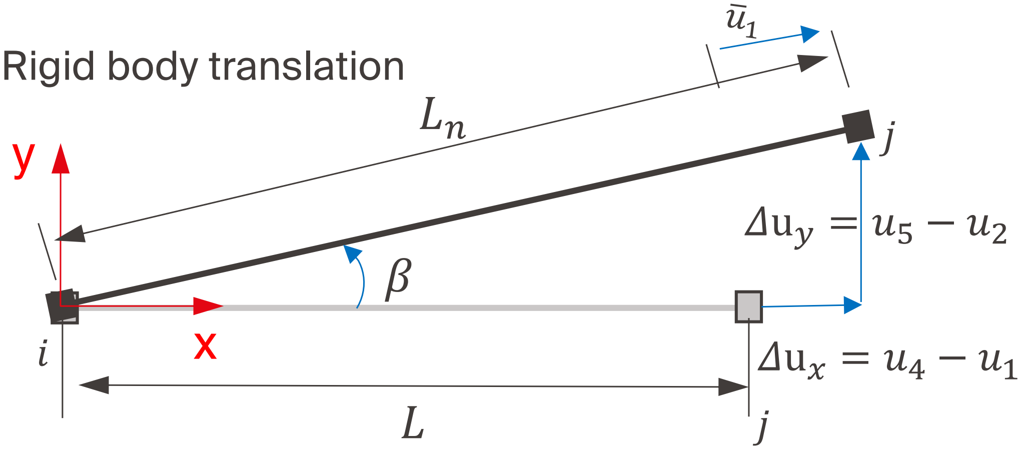


- Conventional frame (beam-column) elements are formulated within the basic reference frame

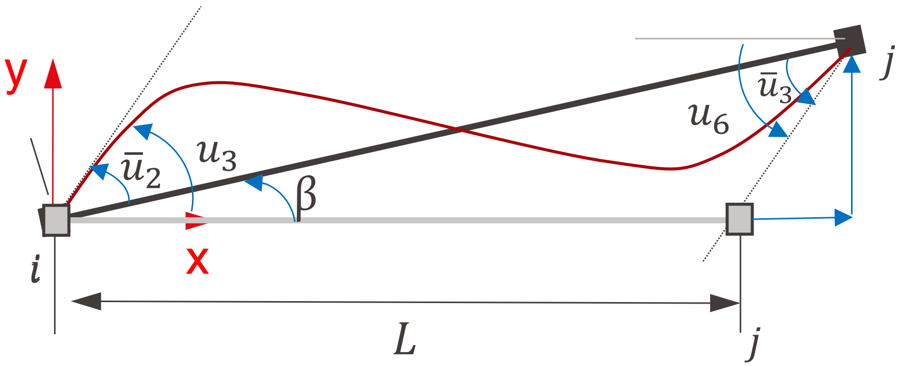
# Removal of rigid body modes



Rigid body translation



Deformations in the basic reference frame



# Displacement–deformation relation: Large displacements

- The displacements in the basic reference frame are given by

$$
\begin{aligned}
\bar{u}_1 &= L_n - L \\
\bar{u}_2 &= u_3 - \beta \\
\bar{u}_3 &= u_6 - \beta
\end{aligned}
$$

With

$$\beta = \arctan\left(\frac{\Delta u_y}{L + \Delta u_x}\right), L_n = \sqrt{(L + \Delta \mathrm{u}_x)^2 + \left(\Delta u_y\right)^2}$$

# Linear geometry approximation

- Assuming small deformations and rotations, expanding the arctan function using a Taylor series approximation about the point $\Delta u_x = 0, \Delta u_y = 0$,

$$\beta = \frac{\Delta u_y}{L}\left[1 - \frac{\Delta u_x}{L} + \cdots\right]$$

Similarly,

$$\bar{u}_1 = L\left[\frac{\Delta u_x}{L} + \frac{1}{2}\left(\frac{\Delta u_x}{L}\right)^2 + \frac{1}{2}\left(\frac{\Delta u_y}{L}\right)^2\right]$$

- Assuming small deformations and rotations, the second order terms can be neglected,

$$\beta = \frac{\Delta u_y}{L}$$

And

$$\bar{u}_1 = \Delta u_x$$

# Linear geometry approximation (2)

- Assuming linear geometry (i.e., small deformations and rotations), the compatibility relations between the element deformations in the basic reference frame $\bar{\mathbf{u}}$ and the element displacements in the local reference frame $\boldsymbol{u}$ become linear:

$$\bar{u}_1 = \Delta u_x = u_4 - u_1 = -1u_1 + 0u_2 + 0u_3 + 1u_4 + 0u_5 + 0u_6$$

$$\bar{u}_2 = u_3 - \frac{\Delta \mathrm{u}_y}{L} = u_3 - \frac{u_5 - u_2}{L} = 0u_1 + \frac{1}{L}u_2 + 1u_3 + 0u_4 - \frac{1}{L}u_5 + 0u_6$$

$$\bar{u}_3 = u_6 - \frac{\Delta \mathrm{u}_y}{L} = u_6 - \frac{u_5 - u_2}{L} = 0u_1 + \frac{1}{L}u_2 + 0u_3 + 0u_4 - \frac{1}{L}u_5 + 1u_6$$

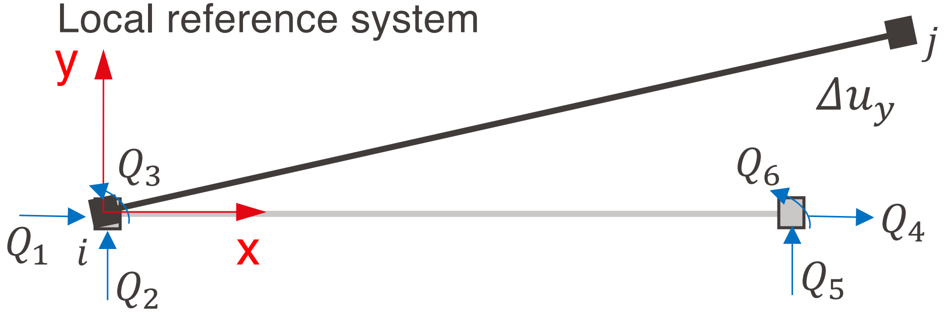
- In matrix form

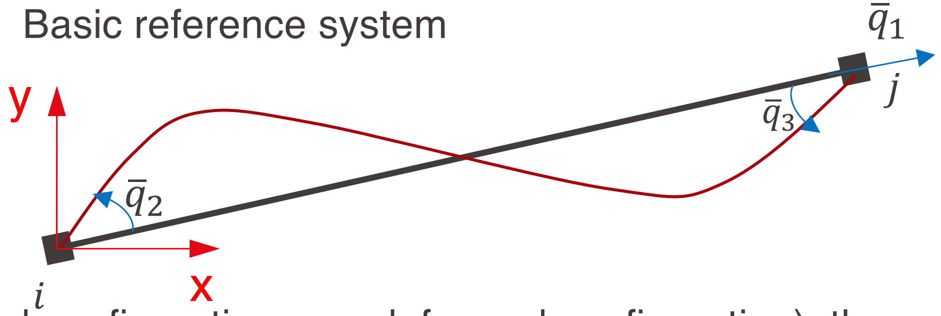
$$\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \end{pmatrix} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{L} & 1 & 0 & -\frac{1}{L} & 0 \\ 0 & \frac{1}{L} & 0 & 0 & -\frac{1}{L} & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}$$

In compact form

$$\bar{\mathbf{u}} = \mathbf{L}\mathbf{u}$$

# Linear geometry approximation (3)


Local reference system


Basic reference system

- Using the equilibrium of forces (linear geometry → deformed configuration ≈ undeformed configuration), the element resisting forces in the local reference system are given by

$$
\begin{aligned}
Q_4 &= \bar{q}_1\\
Q_3 &= \bar{q}_2\\
Q_6 &= \bar{q}_3\\
Q_1 &= -Q_4 = -\bar{q}_1\\
Q_2 &= \frac{1}{L}(Q_3+Q_6) = \frac{1}{L}(\bar{q}_2+\bar{q}_3)\\
Q_5 &= \frac{-1}{L}(Q_3+Q_6) = \frac{-1}{L}(\bar{q}_2+\bar{q}_3)
\end{aligned}
$$

- In matrix form

$$
\begin{pmatrix} Q_1\\ Q_2\\ Q_3\\ Q_4\\ Q_5\\ Q_6 \end{pmatrix} = \begin{bmatrix} -1 & 0 & 0\\ 0 & 1/L & 1/L\\ 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & -1/L & -1/L\\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \bar{q}_1\\ \bar{q}_2\\ \bar{q}_3 \end{pmatrix} \textbf{ or } \mathbf{Q} = \mathbf{L}^T\bar{\mathbf{q}}
$$

# Linear geometry approximation (4)

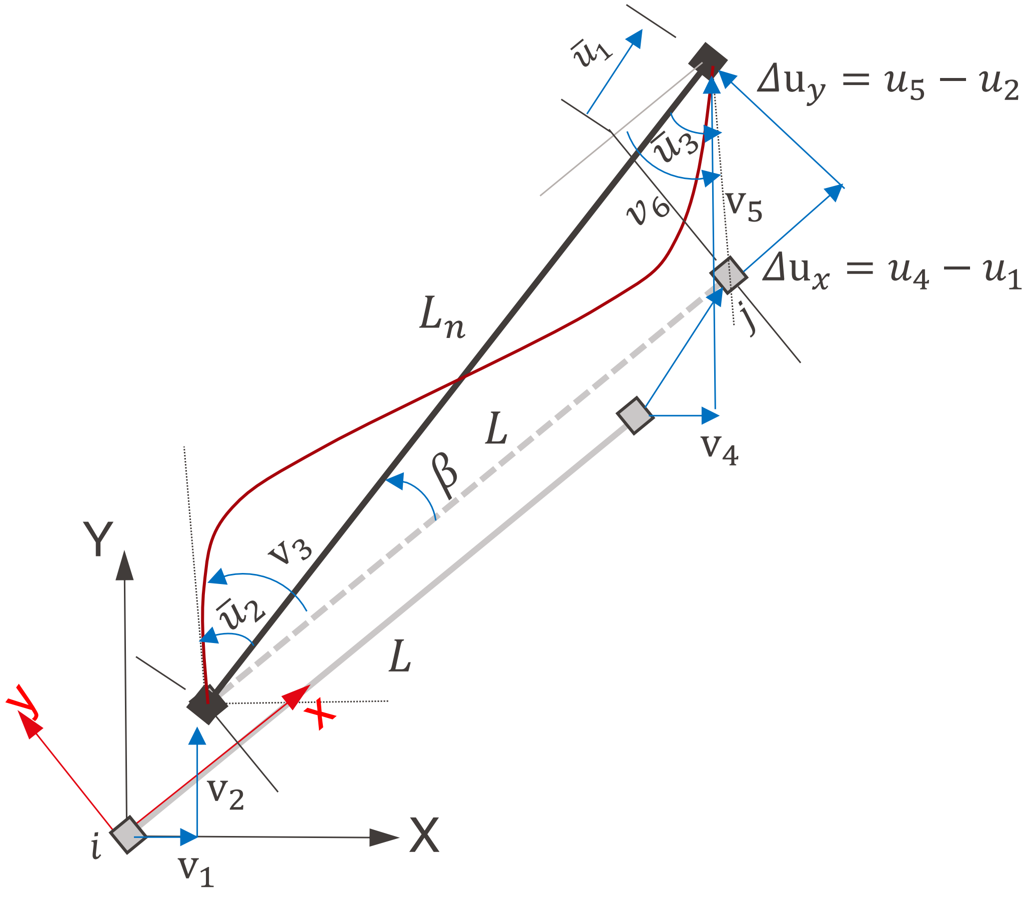
- The element stiffness matrix in the basic reference system, $\bar{\mathbf{k}}$ is given by

$$\bar{\mathbf{k}} = \frac{\partial \bar{\mathbf{q}}}{\partial \bar{\mathbf{u}}}$$

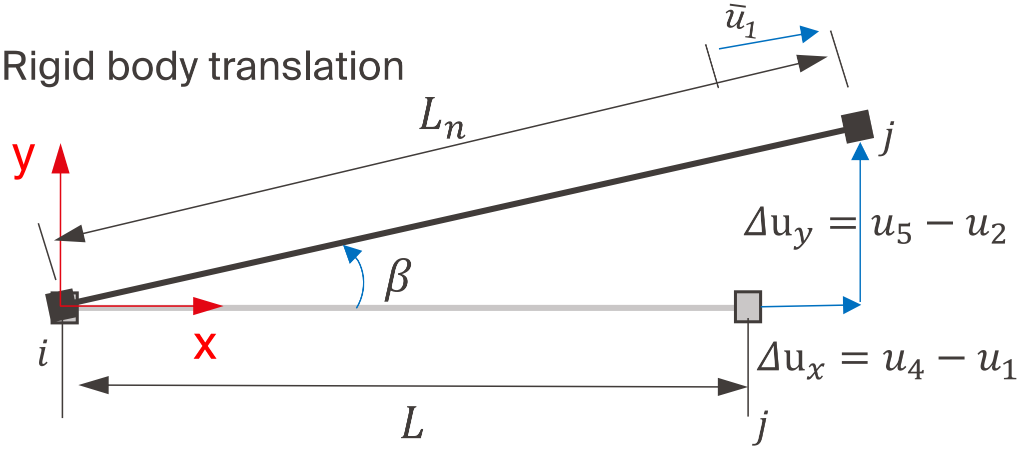
- The element stiffness matrix in the local reference system, $\mathbf{k}_{local}$ is given by

$$\mathbf{k}_{local} = \frac{\partial \mathbf{Q}}{\partial \mathbf{u}} = \frac{\partial}{\partial \mathbf{u}}(\mathbf{L}^T \bar{\mathbf{q}}) = \frac{\partial}{\partial \mathbf{u}}\left(\mathbf{L}^T \bar{\mathbf{k}} \bar{\mathbf{u}}\right) = \frac{\partial}{\partial \mathbf{u}}\left(\mathbf{L}^T \bar{\mathbf{k}} \mathbf{L} \mathbf{u}\right) = \mathbf{L}^T \bar{\mathbf{k}} \mathbf{L}$$

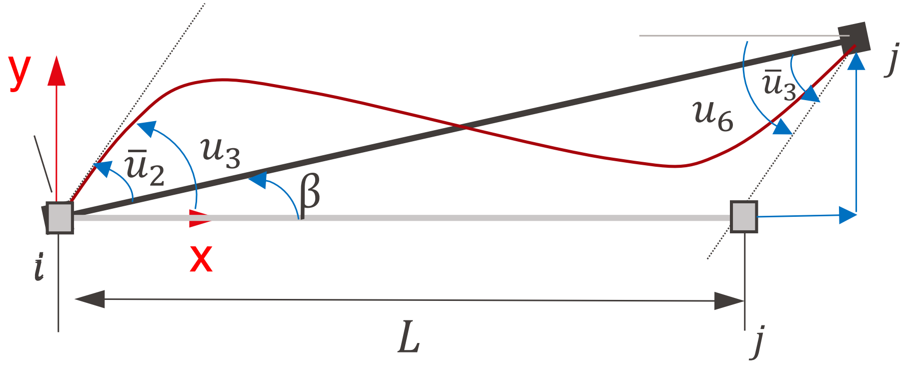
# Nonlinear geometry: Large displacements and rotations



Rigid body translation



Deformations in the basic reference frame



# Displacement–deformation relation: Large displacements

- The displacements in the basic reference frame are given by

$$\bar{u}_1 = L_n - L$$
$$\bar{u}_2 = u_3 - \beta$$
$$\bar{u}_3 = u_6 - \beta$$

With

$$\beta = \arctan\left(\frac{\Delta u_y}{L + \Delta u_x}\right), L_n = \sqrt{(L + \Delta \mathrm{u}_x)^2 + \left(\Delta u_y\right)^2}$$

- Large displacements and rotations → Corotational formulation

- Define

$$c = \cos(\beta) = \frac{L + \Delta u_x}{L_n}, s = \sin(\beta) = \frac{\Delta u_y}{L_n}$$

# Corotational formulation: Virtual displacements

- The virtual basic displacements are obtained through differentiation of the previous equations

$$\delta\bar{u}_1 = \delta L_n = \frac{1}{2L_n}\left(2\delta\Delta u_x(L+\Delta u_x) + 2\Delta u_y \delta\Delta u_y\right) = c(\delta u_4 - \delta u_1) + s(\delta u_5 - \delta u_2)$$
$$= [-c \quad -s \quad 0 \quad c \quad s \quad 0]\delta\mathbf{u}$$

$$\delta\bar{u}_2 = \delta u_3 - \delta\beta$$
$$\delta\bar{u}_3 = \delta u_6 - \delta\beta$$

Where,

$$\delta\beta = \frac{1}{1+\left(\frac{\Delta u_y}{L+\Delta u_x}\right)^2}\left(\frac{\delta\Delta u_y(L+\Delta u_x) - \Delta u_y\delta\Delta u_y}{(L+\Delta u_x)^2}\right)$$

And after simplification

$$\delta\beta = \frac{1}{L_n}[c(\delta u_5 - \delta u_2) - s(\delta u_4 - \delta v_1)] = \frac{1}{L_n}[s \quad -c \quad 0 \quad -s \quad c \quad 0]\delta\mathbf{u}$$

# Corotational formulation: Virtual displacements (2)

- The relations between the displacements in the local reference frame $\mathbf{u}$ and the displacements in the basic reference frame $\bar{\mathbf{u}}$ are given by,

$$\begin{cases} \delta\bar{u}_1 = [-c \quad -s \quad 0 \quad c \quad s \quad 0]\delta\mathbf{u} \\ \delta\bar{u}_2 = \delta u_3 - \dfrac{1}{L_n}[s \quad -c \quad 0 \quad -s \quad c \quad 0]\delta\mathbf{u} \\ \delta\bar{u}_3 = \delta u_6 - \dfrac{1}{L_n}[s \quad -c \quad 0 \quad -s \quad c \quad 0]\delta\mathbf{u} \end{cases}$$

In matrix form:

$$\delta\bar{\mathbf{u}} = \mathbf{L}\delta\mathbf{u}$$

With

$$\mathbf{L} = \begin{bmatrix} -c & -s & 0 & c & s & 0 \\ -s/L_n & c/L_n & 1 & s/L_n & -c/Ln & 0 \\ -s/L_n & c/L_n & 0 & s/L_n & -c/L_n & 1 \end{bmatrix}$$

# Corotational formulation: Resisting forces

- The relation between the resisting forces in the basic reference system $\overline{\mathbf{q}}$ and the resisting forces in the local reference system $\mathbf{Q}$ is obtained by equating the virtual work in both basic and local reference systems,

$$W = \delta\mathbf{u}^T\mathbf{Q} = \delta\overline{\mathbf{u}}^T\overline{\mathbf{q}} = \delta\mathbf{v}^T\mathbf{L}^T\overline{\mathbf{q}}$$

- This equation must hold true for any arbitrary $\delta\mathbf{u}^T$, therefore, the element resisting forces in the local reference frame $\mathbf{Q}$ are given by

$$\mathbf{Q} = \mathbf{L}^T\overline{\mathbf{q}}$$

- The element resisting forces in the basic reference frame $\overline{\mathbf{q}}$ depend on the element formulation and will be discussed in the following weeks.

# Corotational formulation: Tangent stiffness matrix

- The tangent stiffness in the local reference frame $\mathbf{K}_l$ defined by

$$\delta\mathbf{Q} = \mathbf{K}_l \delta\mathbf{u}$$

is obtained by differentiating the relation $\mathbf{Q} = \mathbf{L}^T\overline{\mathbf{q}}$, which gives

$$\delta\mathbf{Q} = \mathbf{L}^T\delta\overline{\mathbf{q}} + \delta\mathbf{L}^T\overline{\mathbf{q}} = \mathbf{L}^T\delta\overline{\mathbf{q}} + \bar{q}_1\delta\mathbf{L}_1 + \bar{q}_2\delta\mathbf{L}_2 + \bar{q}_3\delta\mathbf{L}_3$$

Where $\mathbf{L}_k$ is the $k^{th}$ column of $\mathbf{L}^T$

- Introduce the quantities

$$\mathbf{r} = [-c \quad -s \quad 0 \quad c \quad s \quad 0]^T$$
$$\mathbf{z} = [s \quad -c \quad 0 \quad -s \quad c \quad 0]^T$$

And differentiate them

$$\delta\mathbf{r} = \mathbf{z}\delta\beta, \delta\mathbf{z} = -\mathbf{r}\delta\beta$$

These can be used to rewrite

$$\delta\bar{u}_1 = \delta L_n = \mathbf{r}^T\delta\mathbf{u}, \qquad \delta\beta = \frac{\mathbf{z}^T}{L_n}\delta\mathbf{u}$$

# Corotational formulation: Tangent stiffness matrix (2)

- Using these notations

$$\mathbf{L}_1 = \mathbf{r}$$

$$\mathbf{L}_2 = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^T - \frac{\mathbf{z}}{L_n}$$

$$\mathbf{L}_3 = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T - \frac{\mathbf{z}}{L_n}$$

Which by differentiation give

$$\delta\mathbf{L}_1 = \delta\mathbf{r} = \frac{\mathbf{z}\mathbf{z}^T}{L_n}\delta\mathbf{u}$$

$$\delta\mathbf{L}_2 = \delta\mathbf{L}_3 = -\frac{\delta\mathbf{z}}{L_n} + \frac{\mathbf{z}\delta L_n}{L_n^2} = \frac{1}{L_n^2}(\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T)\delta\mathbf{u}$$

# Corotational formulation: Tangent stiffness matrix (3)

- The tangent stiffness matrix in the basic reference system $\bar{\mathbf{k}}$ defined by

$$\delta\bar{\mathbf{q}} = \bar{\mathbf{k}}\delta\bar{\mathbf{u}} = \bar{\mathbf{k}}\mathbf{L}\delta\mathbf{u}$$

- Finally

$$\begin{aligned}\delta\mathbf{Q} = \mathbf{K}_l\delta\mathbf{u} &= \mathbf{L}^T\delta\bar{\mathbf{q}} + \bar{q}_1\delta\mathbf{L}_1 + \bar{q}_2\delta\mathbf{L}_2 + \bar{q}_3\delta\mathbf{L}_3 \\ &= \mathbf{L}^T\bar{\mathbf{k}}\mathbf{L}\delta\mathbf{u} + \bar{q}_1\frac{\mathbf{z}\mathbf{z}^T}{L_n}\delta\mathbf{u} + \bar{q}_2\frac{1}{L_n^2}(\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T)\delta\mathbf{u} + \bar{q}_3\frac{1}{L_n^2}(\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T)\delta\mathbf{u} \\ &= \left(\mathbf{L}^T\bar{\mathbf{k}}\mathbf{L} + \bar{q}_1\frac{\mathbf{z}\mathbf{z}^T}{L_n} + \bar{q}_2\frac{1}{L_n^2}(\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T) + \bar{q}_3\frac{1}{L_n^2}(\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T)\right)\delta\mathbf{u}\end{aligned}$$

- The tangent stiffness in the local reference frame $\mathbf{K}_l$ is given by

$$\mathbf{K}_l = \underbrace{\mathbf{L}^T\bar{\mathbf{k}}\mathbf{L}}_{\text{Material stiffness matrix}} + \underbrace{\bar{q}_1\frac{\mathbf{z}\mathbf{z}^T}{L_n} + \frac{1}{L_n^2}(\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T)(\bar{q}_2 + \bar{q}_3)}_{\text{Geometric stiffness matrix}}$$

- The element tangent stiffness matrix in the basic reference frame $\bar{\mathbf{k}}$ depends on the element formulation. This will be discussed in the coming weeks.

# Corotational formulation: Geometric stiffness matrix

- The geometric stiffness matrix $\mathbf{k}_{geom}$ is computed using

$$\mathbf{k}_{geom} = \bar{q}_1 \frac{\mathbf{z}\mathbf{z}^T}{L_n} + \frac{1}{L_n^2}\left(\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T\right)(\bar{q}_2 + \bar{q}_3)$$

$$= \frac{\bar{q}_1}{L_n}\begin{bmatrix} s^2 & -cs & 0 & -s^2 & cs & 0 \\ -cs & c^2 & 0 & cs & -c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2 & cs & 0 & s^2 & -cs & 0 \\ cs & -c^2 & 0 & -cs & c^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\bar{q}_2 + \bar{q}_3}{L_n^2}\begin{bmatrix} -2sc & c^2 - s^2 & 0 & 2sc & -c^2 + s^2 & 0 \\ c^2 - s^2 & 2cs & 0 & -c^2 + s^2 & -2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2sc & -c^2 + s^2 & 0 & -2sc & c^2 - s^2 & 0 \\ -c^2 + s^2 & -2cs & 0 & c^2 - s^2 & 2cs & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Transformation to the global reference system

- From the previous lectures, to transform the stiffness matrix from the local reference frame $\mathbf{K}_l$ to the stiffness matrix in the global reference system $\mathbf{K}_g$, the transformation matrix $\mathbf{T}$ is used:

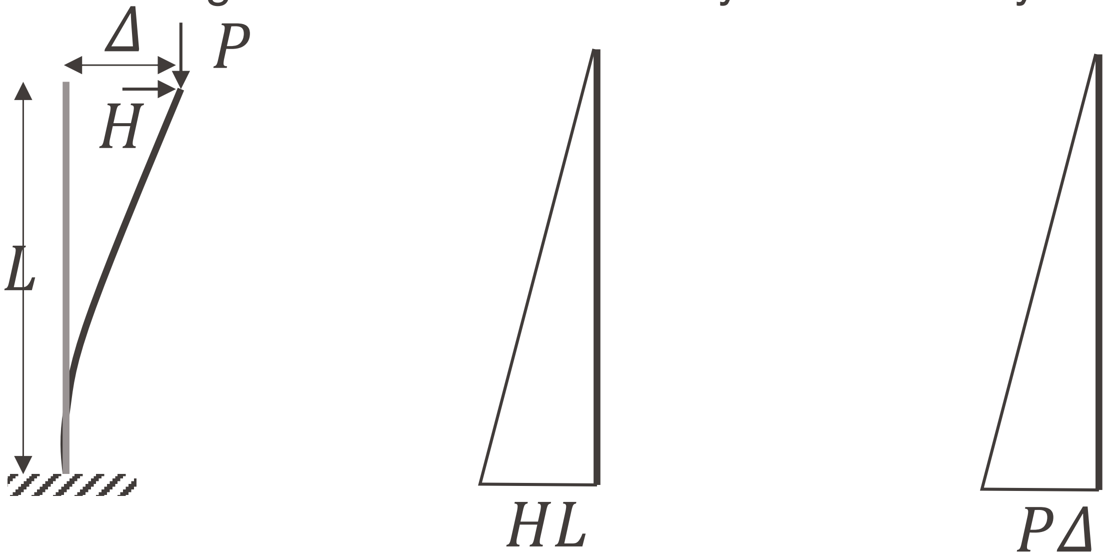
$$\mathbf{K}_g = \mathbf{T}^T\mathbf{K}_l\mathbf{T}$$

- Using the definition of the local stiffness matrix $\mathbf{K}_l$ using the stiffness matrix from the basic reference frame $\bar{\mathbf{k}}$ gives

$$\mathbf{K}_g = \underbrace{\mathbf{T}^T\left(\underbrace{\underbrace{\mathbf{L}^T\bar{\mathbf{k}}\mathbf{L}}_{\text{Material stiffness matrix}} + \underbrace{\bar{q}_1\frac{\mathbf{z}\mathbf{z}^T}{L_n} + \frac{1}{L_n^2}(\mathbf{r}\mathbf{z}^T + \mathbf{z}\mathbf{r}^T)(\bar{q}_2 + \bar{q}_3)}_{\text{Geometric stiffness matrix}}}_{\text{Tangent stiffness matrix in the local reference system}}\right)\mathbf{T}}_{\text{Tangent stiffness matrix in the global reference system}}$$

# P-Delta effects on columns

- Consider the following column loaded laterally and vertically



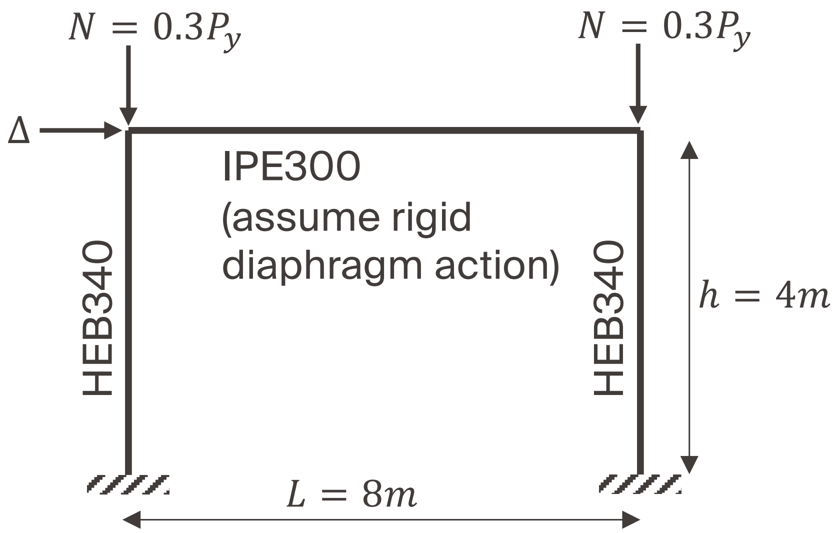
$1^{st}$ order moment
(Linear geometry)

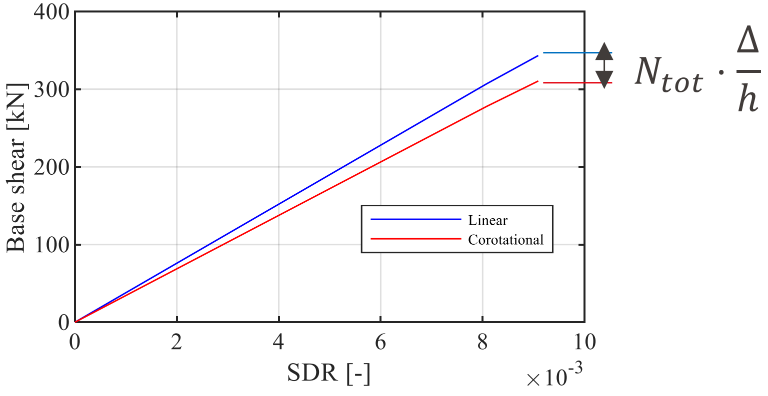
$2^{st}$ order moment
(Nonlinear geometry)

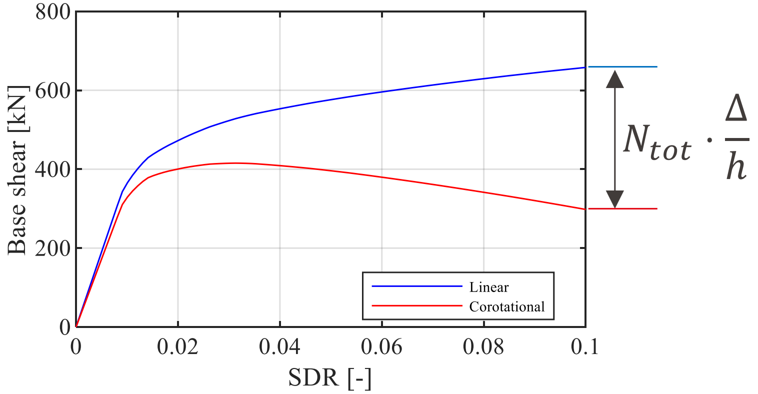
- Assume linear geometry: P-Delta effects are not captured
- Assuming a linear versus nonlinear geometry strongly influences the simulation results

# Nonlinear geometric effects on the response of frames

- Assuming a linear or nonlinear geometry is very important when modeling frame structures.







- For the simulations, a 2d force-based beam-column element was used and the UVC (i.e., isotropic hardening rule) constitutive material law. These will be explained later.