Traffic Engineering (CIVIL-349)
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Exercise 3 - Solution
Input-Output diagrams & LWR Theory
Fall 2025

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Problem 1

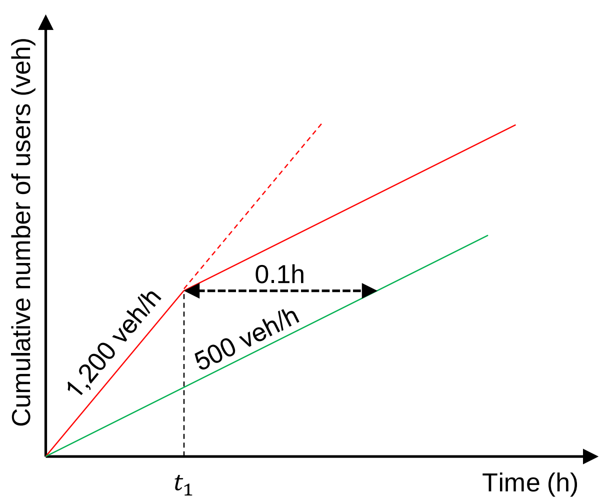
a. This first question requires solving the dynamics of queues for the entire 4 hours period considered. Let w_1 denote the waiting time on link 1 when there is a queue at the bottleneck. Wardrop's principle implies that users choose link 2 if and only if

$$w_1 + \tau_1 >= \tau_2 \tag{1}$$

i.e. $w_1 >= 0.1h$

Note that since there is no capacity constraint on link 2, the delay at the bottleneck link 1 cannot exceed 0.1 h. Indeed, as soon as it is slightly greater than 0.1h, all users would choose link 2 and the queue would return to 0.1h. Hence, whenever the arrival flow is greater than the capacity of the bottleneck for some period, the queue builds up until it reaches the value of 0.1 and then remains in this state, which is an equilibrium.

Let t_1 denote the time at which the waiting time reaches this equilibrium value.



The input-output diagram for the bottleneck at the beginning of link 1 allows us to find t_1 easily. Before t_1, the delay per user is shorter than 0.1h so all the incoming vehicles choose link 1 (1, 200 veh/h). The number of vehicles that have arrived at the bottleneck at t_1 is given by $1,200t_1$. Because the queue has the First-In-First-Out property, this same number of vehicles should have passed the bottleneck when $t = t_1 + 0.1$, i.e. when the $1,200t_1$-th user has waited for 0.1 hours. Hence

$$1200t_1 = 500(t_1 + 0.1) \tag{2}$$

So $t_1 = \frac{50}{700} \approx 0.0714$h.

Since $t_1 < 1$, the queue has time to reach its equilibrium length and then remains constant as long as the demand is greater than the capacity of link 1, i.e. during $1 - t_1$ hours.

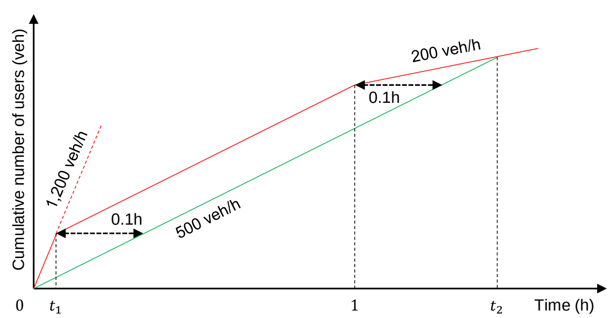
After $t = 1$ h, the queue starts emptying because even the total inflow (200 veh/h) cannot compensate for the outflow at the bottleneck (500 veh/h). Hence, all users choose link 1 again.

Overall, users choose link 2 only between t_1 and 1. Since the queue for link 1 remains constant during that time, the inflow in link 1 is equal to the capacity, i.e. 500 veh/h. The rest goes to link 2. In total,$N_2 = (1 - t_1)(1200 - 500) = 650$ vehicles choose link 2.

b. The total travel time can be expressed as the sum of three terms:

$$TTT = \text{Total delay at the bottleneck} + 0.2N_2 + 0.1N_1 \tag{3}$$

We know that $N_2 = 650$ vehicles and that $N_1 = 1200 + 3 \times 200 - N_2 = 1150$ vehicles. The only "complicated" term is the total delay at the bottleneck.



Since the horizontal distance between the input curve (in red) and the output curve (in green) is the delay for one user, the total delay is simply the area of the quadrilateral between the red and the green curves. It is easiest to compute is separately for three types of users:

- There are $1,200t_1$ users arriving at the bottleneck between 0 and t_1, experiencing an average delay of 0.05h (the delays are actually uniformly distributed between 0 and 0.1h).

- There are $(1 - t_1) \times 500$ users arriving between t_1 and $t = 1$, experiencing a delay of 0.1h.
- There are $200(t_2 - 1)$ users arriving at the bottleneck between $t = 1$ and t_2, experiencing an average delay of 0.05h.

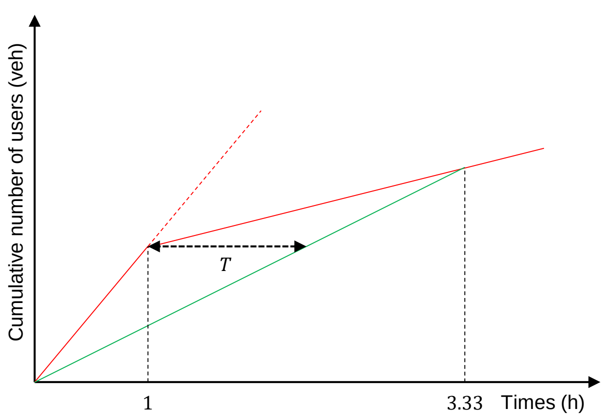
t_2 is found like t_1, by writing that $200(t_2 - 1) = 500(t_2 - (0.1 + 1))$, i.e. $t_2 = \frac{350}{300} \approx 1.167$h.

Summing all these delays yields a total delay of 52.38 veh.h, and a total travel time of 297.38 veh.h.

c. If link 2 is closed, the calculations are much easier. The queue length increases as long as the demand is greater than the capacity, i.e. for 1h. After 1h, 1,200 users have arrived at the bottleneck, but only 500 have passed, so the maximum queue length is 700. Afterwards, the queue length changes at a rate of $200 - 500 = -300$ veh/h (inflow minus outflow). Hence, it takes $700/300 = 2.33$ hours to empty the queue after the end of the peak.

The maximum waiting time T is given by $1200 \times 1 = 500 \times (1 + T)$, i.e. $T = 1.4$.

Hence, in total $1 \times 1200 + 2.33 * 200$ vehicles have been delayed at the bottleneck and they experienced an average delay of $T/2 = 0.7$h. The total delay is therefore 1,167 veh.h. The total free-flow travel time is $(1200 + 3 \times 200) \times 0.1 = 180$ veh.h. $\Rightarrow$ Total travel time $= 1,347$ veh.h. Again, an input-output diagram helps a lot.



Problem 2

a. At first, we draw the triangular flow-density Fundamental Diagram that describes traffic in this freeway segment, which can be seen in Figure 1. Since there are 3 lanes, the total capacity of the freeway (critical flow) is $q_{\text{cr}} = 2000 \times 3 = 6000$ veh/hr. The critical density is calculated as:

$$k_{\text{cr}} = \frac{q_{\text{cr}}}{v_{\text{ff}}} = \frac{6000 \text{ veh/h}}{100 \text{ km/h}} = 60 \text{ veh/km}$$

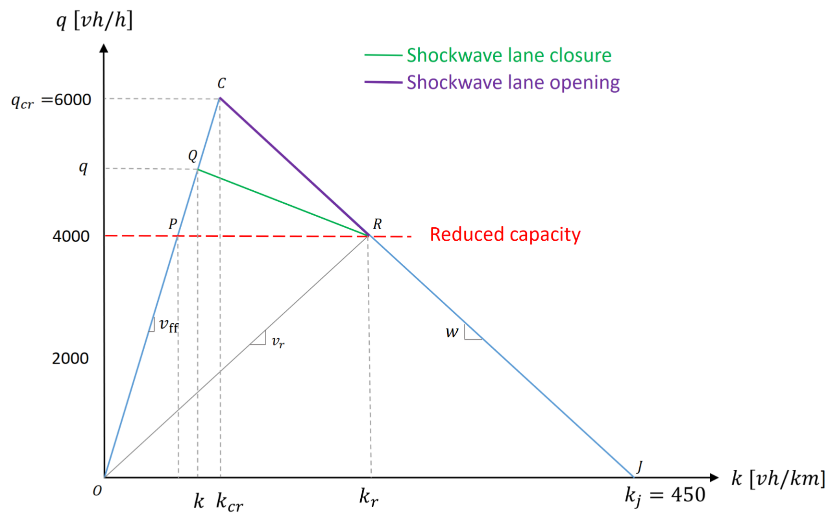


Figure 1: Triangular Fundamental Diagram for the 3-lane road segment.

At the initial state, the incoming flow is $q = 5000$ veh/h. Since this is lower than the critical flow, vehicles travel with free flow speed $v_{\text{ff}} = 100$ km/h (traffic is in the uncongested part of the FD). Then, from the fundamental relationship $q = kv$, we calculate the initial density:

$$k = \frac{q}{v_{\text{ff}}} = \frac{5000}{100} = 50 \text{ veh/km} \tag{4}$$

When the accident occurs, one lane of the freeway gets closed and a bottleneck is formed at point A (see figure in the description of the exercise). This results in a drop in the freeway capacity (critical flow) at $q_r = 4000$ veh/h.

As we see in Figure 1, once the capacity is reduced, and given that the initial density k is already higher than the critical density that corresponds to the reduced capacity ($\frac{4000}{100} = 40$ veh/km), traffic state at this point will move towards the congested part of the FD, i.e. from point Q (lower density state) to point R (higher density state), resulting in a rapid increase of the density and in the formation of a shockwave. The speed of vehicles will also decrease from v_{ff} to v_r. According to LWR theory, the shockwave will propagate backwards with a speed that is equal to the slope of the line QR in Figure 1, (ratio of change in flow over change in density).

In order to find the slope of QR, we first need to calculate the slope w of line CJ, which represents the speed of the shockwave when the lane reopens, i.e. when traffic moves from state R (or J) of higher density to state C of lower density. This slope can be found as:

$$w = \frac{q_{\text{cr}} - q_j}{k_{\text{cr}} - k_j} = \frac{6000 - 0}{60 - 450} = -15.38 \text{ km/h} \tag{5}$$

Then, by using w we can calculate density at point R as follows:

$$w = \frac{q_r - q_j}{k_r - k_j} \Leftrightarrow k_r = \frac{q_r - q_j}{w} + k_j = \frac{4000}{-15.38} + 450 \simeq 190 \text{ veh/km} \tag{6}$$

Finally, the speed of the shockwave formation can be found by calculating the slope of line QR, as follows:

$$\text{Slope QR} = u = \frac{q_r - q}{k_r - k} = \frac{4000 - 5000}{190 - 50} \simeq -7.14 \text{ km/h} \tag{7}$$

Hence, due to the closure of one lane at 9.45 am, a queue of vehicles starts forming upstream of the point of the accident, which propagates backwards with a speed of 7.14 km/h.

Also, based on the above calculations, we can also find the speed of the vehicles at the state of reduced capacity R, as:

$$v_r = \frac{q_r}{k_r} = \frac{4000}{190} = 21.05 \text{ km/h} \tag{8}$$

Based on all the above information, we can draw the time-space diagram to depict the movement of vehicles, before, during and after the lane closure. The time-space diagram is shown in Figure 2. The formation of the shockwave due to the lane closure (transition $Q \rightarrow R$) is drawn in green color, while the rapid dissolution of the queue of vehicles due to lane reopening creates a shockwave drawn in purple (transition $R \rightarrow C$). The slopes of these shockwaves are the ones that we calculate from equations 7 and 5, respectively.

Each distinct traffic state that is observed on the time-space diagram is assigned to the states of different points of the Fundamental Diagram, as we can see in Figure 2. The speed in states Q, P and C is the free flow speed v_{ff}, while the speed in state R is v_r, which is calculated in equation 8. Knowing the flow and density in each state, we can compute the average headway and spacing, respectively, at each different state, so that the trajectories of vehicles can be drawn.

b. We can calculate the time that Lucy will pass by the accident scene by using the time-space diagram of Figure 2. We are interested

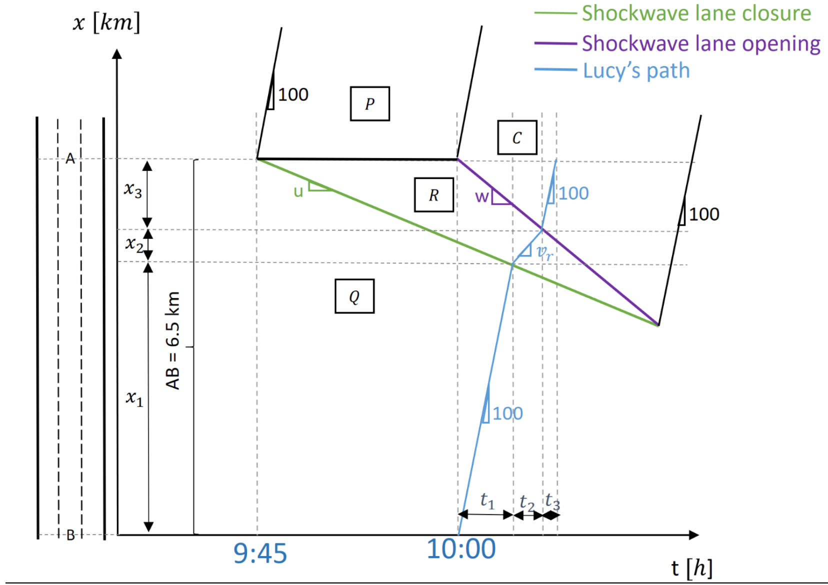


Figure 2: Time-Space diagram describing traffic flow before and after the accident. We can see the shockwave generation (due to lane closure) and dissipation (after lane reopening).

in calculating the periods t_1, t_2 and t_3 that Lucy needs to cover distances x_1, x_2 and x_3, respectively, until she reaches the point of the accident. By applying basic geometry, we can write:

$$x_1 = 100t_1 \quad \text{(km)} \tag{9}$$

$$x_1 = 6.5 - |u|\left([10.00\text{am} - 9.45\text{am}] + t_1\right) = 6.5 - 7.14(0.25 + t_1) \tag{10}$$

By equations 9 and 10 we find that $x_1 = 4.4$ km and $t_1 = 0.044$ h.
Similarly, for distance x_2 we can write:

$$x_2 = v_r t_2 = 21.05 \times t_2 \tag{11}$$

$$x_2 = 6.5 - x_1 - |w|(t_1 + t_2) = 2.1 - 15.38(0.044 + t_2) \tag{12}$$

By equations 11 and 12 we find that $x_2 = 0.821$ km and $t_2 = 0.039$ h.

Then $x_3 = 6.5 - x_1 - x_2 = 1.28$ km and $x_3 = 100t_3$, so $t_3 = 0.0128$ h. Given that Lucy is at point B at 10.00 am, she will need another $t = t_1 + t_2 + t_3 = 0.096$ h $= 5$ min and 45 sec to reach point A. Hence, she will be at point A at 10 : 05 : 45 am.