CIVIL-312: Hydraulic Engineering and Infrastructures

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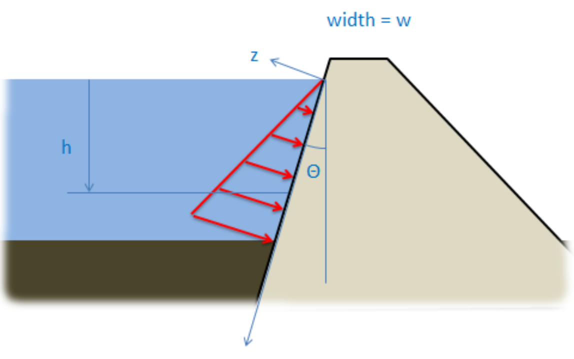
Last week recap

- We introduced the course and went over the logistics
- Reviewed the derivation of Reynolds Transport Theorem
- Reviewed the Continuity Equation derivation and application
- Reviewed the Energy Equation derivation and application
- Reviewed the Momentum Equation derivation and application

Any question regarding the logistics
or the review?



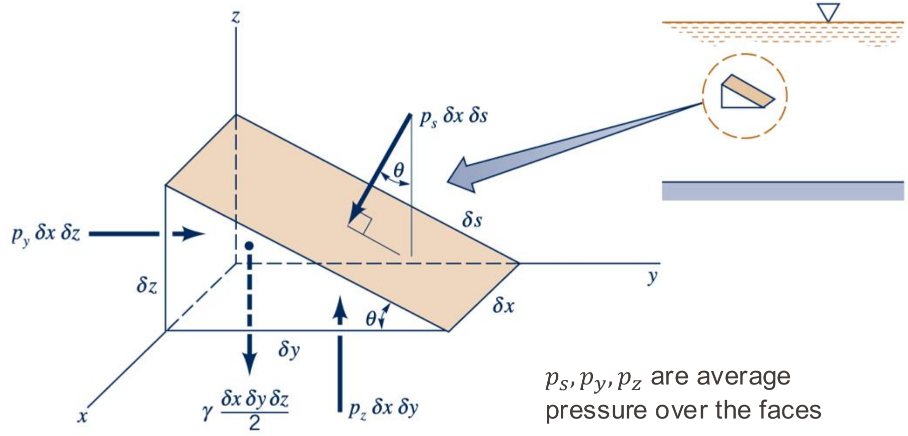
Hydrostatics

This week

- Review of Pascal Law
- Review of Hydrostatic Law
- Review of forces on a flat surface – magnitude and application point
- Pressure prism
- Review of forces on a curved surface

Pascal Law

In quiescent water (static, V=0), the forces acting on a surface are only normal to the surface and tangential forces are 0. This implies that the pressure at a point is equal in all directions! → PASCAL LAW



p_s, p_y, p_z are average pressure over the faces

1. The water is quiescent, only normal stresses
2. The only external forces acting on the element are pressure and weight
3. For a very tiny wedge, we can neglect the weight ($\delta x \delta y \delta z \rightarrow 0$)
4. The example here does not show x direction but it would be the same

No motion, so the sum of the forces must be qual to zero ➡

$$\boxed{\sum F_{ext} = ma = 0}$$

Pascal Law

$$\sum F_{ext} = ma = 0$$

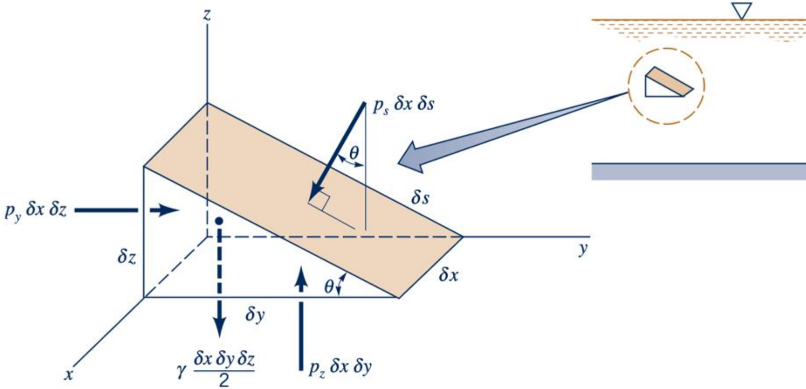
$$\sum F_y = p_y \delta x \delta z - p_s \delta x \delta s \ sin\theta = 0$$

$$\sum F_z = p_z \delta x \delta y - p_s \delta x \delta s \ cos\theta = 0$$

by geometry:

$$\delta s \ sin\theta = \delta z$$

$$\delta s \ cos\theta = \delta y$$



$$p_y \cancel{\delta x}\delta z - p_s \cancel{\delta x}\delta z = 0$$

$$p_z \cancel{\delta x}\delta y - p_s \cancel{\delta x}\delta y = 0$$

$$p_y = p_s$$

$$p_z = p_s$$

The angle θ was arbitrary, so we could use the rationale on any surface at any orientation. Therefore, we can conclude that the pressure at a point in a fluid at rest is independent of the direction, as long as there are no sheering stresses!

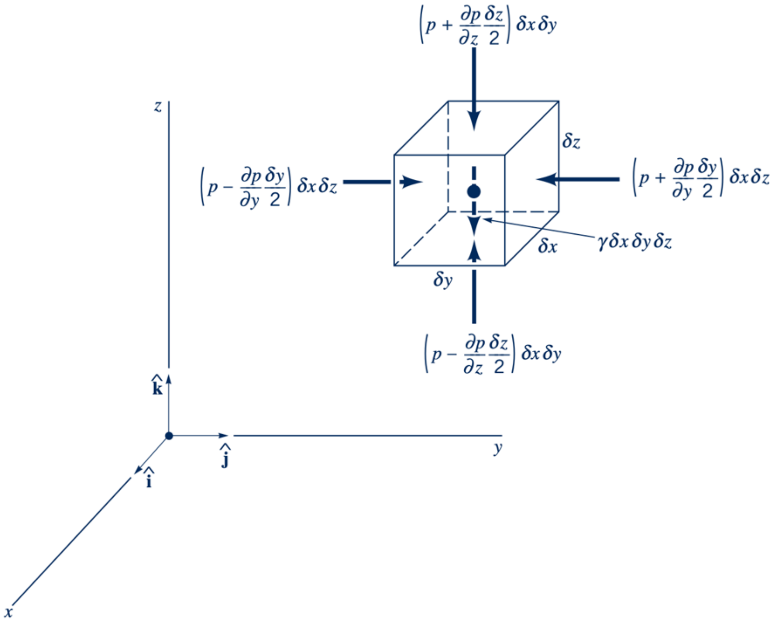
Basic equation for pressure field

We saw that P at a point is independent of direction; but how does P vary from point to point?

Let’s consider a small rectangular element of fluid removed fror some arbitrary position within the mass of the fluid. To understand the force distribution we need to consider all the forces acting on this element:

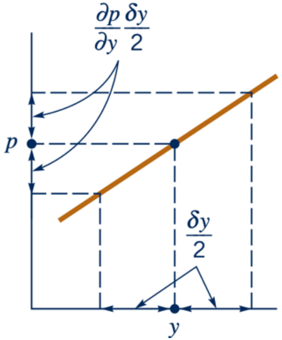
1) SURFACE FORCES due to P (perpendicular to the faces)

2) BODY forces, i.e., the weight

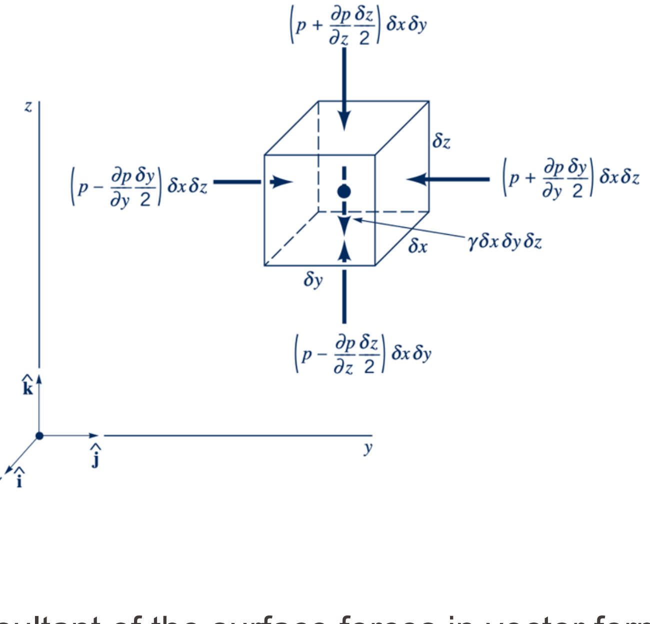


Basic equation for pressure field

Let' start with the surface forces.
Let's assume that the pressure P is at the center of the element and so the average pressure on various faces are small variation of that → "small variation" = derivatives!



It's like expanding P in Taylor series and neglecting the higher order terms



So for example, in the y-direction:

$$\delta F_y=\left(p-\frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x\delta z-\left(p+\frac{\partial p}{\partial y}\frac{\delta y}{2}\right)\delta x\delta z=-\frac{\partial p}{\partial y}\delta x\delta y\delta z$$

Same for the other directions:

$$\delta F_x=-\frac{\partial p}{\partial x}\delta x\delta y\delta z$$

$$\delta F_z=-\frac{\partial p}{\partial z}\delta x\delta y\delta z$$

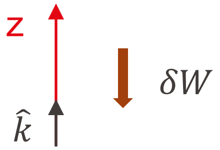
Resultant of the surface forces in vector form

$$\delta F_s=\delta F_x\hat{\imath}+\delta F_y\hat{\jmath}+\delta F_z\hat{k}$$

$$=-\underbrace{\left(\frac{\partial p}{\partial x}\hat{\imath}+\frac{\partial p}{\partial y}\hat{\jmath}+\frac{\partial p}{\partial z}\hat{k}\right)}_{\nabla P}\delta x\delta y\delta z$$

Basic equation for pressure field

Now the body forces: the weight, which is only vertical.



$$-\delta W\hat{k} = -g * \rho * Vol = -\gamma\ \delta x\delta y\delta z\ \hat{k}$$

Sum of the forces must be zero in quiescent water:

$$\sum\delta F = 0 \ \Rightarrow\ \delta F_s - \delta W\hat{k} = 0 \ \Rightarrow\ -\nabla P\delta x\delta y\delta z - \gamma\delta x\delta y\delta z\hat{k} = 0 \ \Rightarrow\ \boxed{\nabla P = -\gamma\hat{k}}$$

$$\nabla P = -\gamma\hat{k} \quad \begin{cases} \dfrac{\partial p}{\partial x} = 0 \\[2ex] \dfrac{\partial p}{\partial y} = 0 \\[2ex] \dfrac{\partial p}{\partial z} = \gamma \end{cases}$$

This shows that the pressure do not depend on x and y but only on z! So, you can move on a horizontal plane at the same z, and P will be the same!

Since P only depend on z, the partial derivative can become total

$$\boxed{\frac{dp}{dz} = -\gamma}$$

Hydrostatic Law

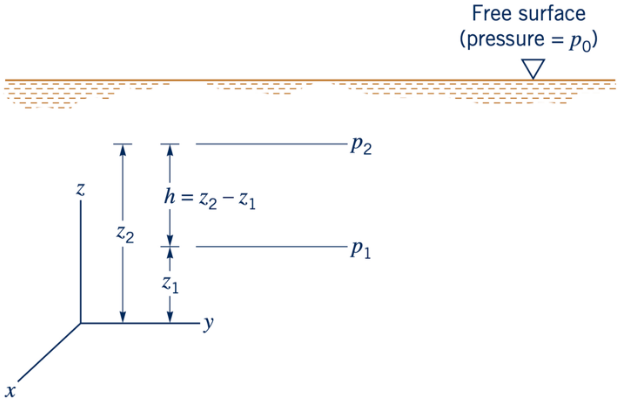
Basic equation for pressure field

Specific weight

$\gamma = \rho g$

Changes in gravity g are negligible in most engineering applications

For **incompressible fluid** (such as water), the density ρ is also constant!



$$\frac{dp}{dz} = -\gamma$$

$$\int_{p_1}^{p_2} dp = -\gamma \int_{z_1}^{z_2} dz \longrightarrow p_1 - p_2 = \gamma(z_2 - z_1)$$

$$\boxed{p_1 = \gamma h + p_2}$$

Hydrostatic distribution

For incompressible fluids at rest, the **pressure varies linearly with depth**!
Pressure increase with depth to “hold up” the fluid above it.

Typically the plane of the water surface (“free surface”) is used as reference and it’s p_{atm} or p_0, so:

$$\boxed{p = \gamma h + p_0}$$

Basic equation for pressure field

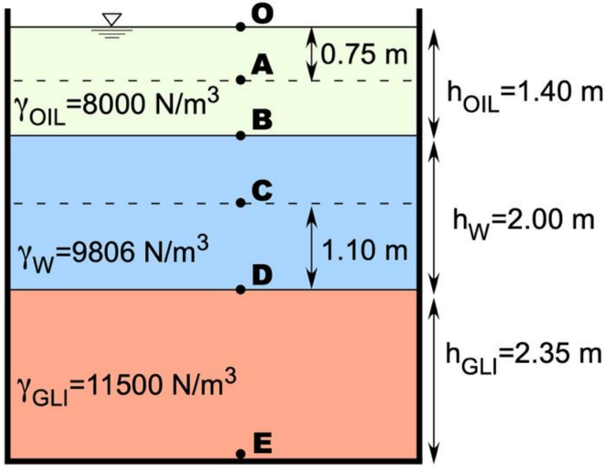
We can also integrate indefinitely:

$$\frac{dp}{dz} = -\gamma \qquad p = -\gamma z + constant$$

$$\boxed{\left(\frac{p}{\gamma} + z\right) = constant}$$

Piezometric head!

Example



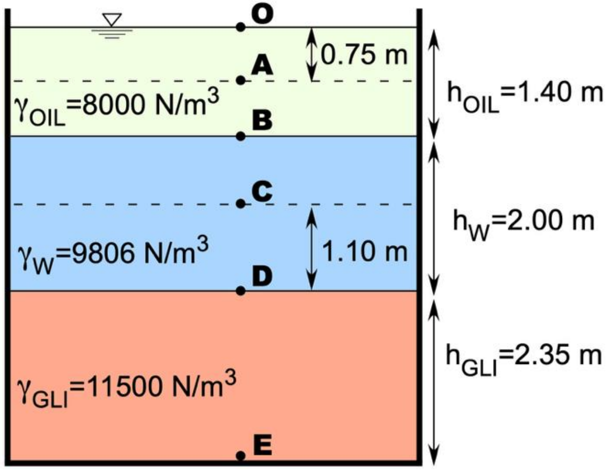
A tank is filled with three liquids with different specific weights, namely oil, water, and glycerin. These fluids are at rest and have the following specific weights:

$$\gamma_{OIL} = 8000\frac{N}{m^3};\ \gamma_W = 9806\frac{N}{m^3};\ \gamma_{GLI} = 11500\frac{N}{m^3}.$$

Calculate:

- the pressure at point **A**;
- the pressure at the oil-water interface (point **B**);
- the pressure at point **C**;
- the pressure at the water-glycerin interface (point **D**);
- the pressure at the bottom of the tank (point **E**);
- draw the diagram of the pressures acting along the wall of the tank.

Example



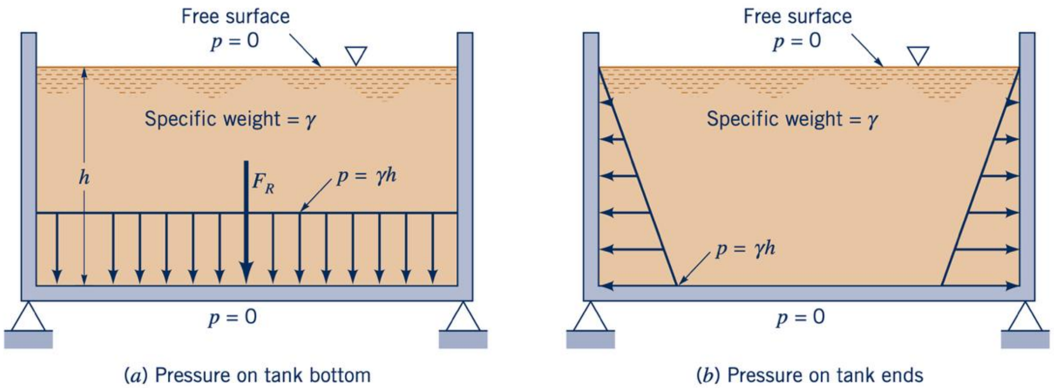
Resolution in class together and solution in the PDF manual online

Hydrostatic forces

Determination of forces due to fluids is essential in design of submerged structures!

What do we know so far?

1) for fluids at rest, the force is perpendicular to a surface (no shearing) and equal from whatever direction
2) pressure varies linearly with depth



(*a*) Pressure on tank bottom

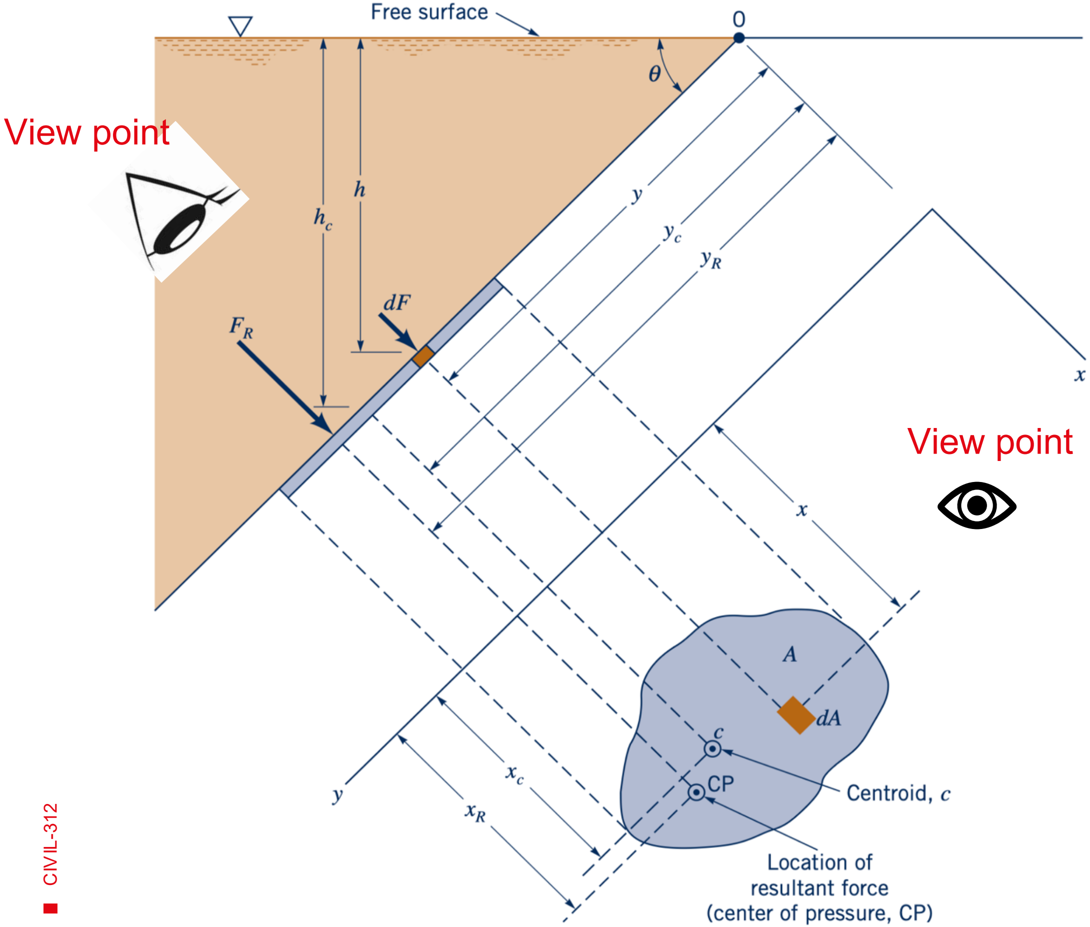
(*b*) Pressure on tank ends

At the bottom, pressure is uniformly distributed so the resulting force is simply:

$$F_R = P * A = \gamma h A$$

On the lateral walls pressure is NOT uniformly distributed.
If the wall is vertical is relatively easy to calculate the Force and its application point.
But let’s look at a generic case: let’s calculate the resulting force on a random surface at a random inclination in the fluid.

Hydrostatic forces on a plane surface



Note:

1. Let the plane in which the surface of A lies intersect the free surface in 0 at a random angle θ
2. The plane x-y of the face A has origin in 0 and is directed along the surface
3. We want to find the resultant force F_R which applies in the center of pressure CP which is different form the centroid of the surface

Hydrostatic forces on a plane surface

(Derivation by hand at the board – summary here in slides)

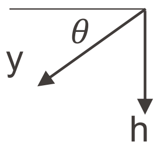
Let’s take a small part of A, namely δA, at a depth h. The small force is: $dF = p\,dA = \gamma h\,dA$

So the magnitude of resultant force F_R is the sum of all these differential forces:

$$F_R = \int_A \gamma h\,dA = \int_A \gamma\, y\, sin\theta\,dA = \gamma sin\theta \boxed{\int_A y\,dA}$$

First moment of the area with respect to x-axis and it’ related to the centroid:

$$y_C = \frac{1}{A}\int_A y\,dA$$



h=y sinθ

* Easier to translate direction y since δA is in the plane x-y

$$F_R = \gamma sin\theta\, y_C A = \boxed{\gamma h_C A}$$

h_C is the distance from the surface to the centroid. Therefore, the magnitude of F_R does not depend on the angle θ but only on the weight of the fluid (γ), the total area A, and the depth of the centroid of the area below the surface.

Hydrostatic forces on a plane surface

The coordinates of the force application x_R and y_R (i.e., the center of pressure) must be calculated through the summation of the moments around the axes. In other words, the moment of the resultant force has to be equal to the moment of the distributed force.

$$F_R y_R = \int_A y \, dF$$

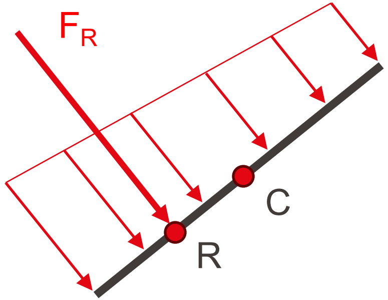
$$\gamma A \, y_C \, sin\theta \; y_R = \int_A y \, \gamma \, y \, sin\theta \, dA$$

$$\cancel{\gamma} A \, y_C \cancel{sin\theta} \; y_R = \cancel{\gamma} \cancel{sin\theta} \int_A y^2 \, dA$$

$$y_R = \frac{1}{y_C A} \boxed{\int_A y^2 \, dA}$$

Moment of inertia I_x with respect to x-axis

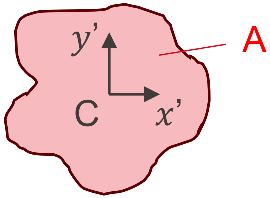
Easy to see that as the pressure increases with depth, so the resultant force is "pushed" below.



Hydrostatic forces on a plane surface

We can express I_x using the moment of inertia of the object with respect to the axes centered in the centroid of the object using the parallel-axis theorem as:

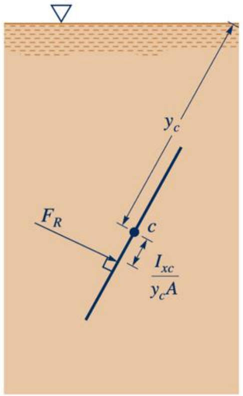
$$I_x = I_{x_C} + A y_C^2$$



(see demonstration of parallel-axis theorem in handwritten notes)

I_{x_C} is the moment of inertia of the object with respect to the axis x' which is parallel to x and passing through C. This simplifies things because I_{x_C} is a property of the object and for regular shapes (rectangular, circular, etc.) are known and tabulated (see after).

$$\Rightarrow \quad y_R = \frac{1}{y_C A}\left(I_{x_C} + A y_C^2\right) \quad \Rightarrow \quad \boxed{y_R = \frac{I_{x_C}}{y_C A} + y_C}$$



Hydrostatic forces on a plane surface

Similarly, x_R is determined summing moments about the y-axis

$$F_R x_R = \int_A x\,dF = \int_A \gamma sin\theta\; xy\,dA \Rightarrow x_R = \frac{\int_A xy\,dA}{y_C A} = \frac{I_{xy}}{y_C A}$$

Similarly to before:
$I_{xy} = \mathrm{I}_{xy_C} + \mathrm{A}x_C y_C$

$$\Downarrow$$

$$\boxed{x_R = \frac{I_{xy_C}}{y_C A} + x_C}$$

So, ultimately:

$$\boxed{\begin{aligned} y_R - y_C &= \frac{I_{x_C}}{y_C A} \\ x_R - x_C &= \frac{I_{xy_C}}{y_C A} \end{aligned}}$$

Hydrostatic forces on a plane surface

Symmetry considerations:

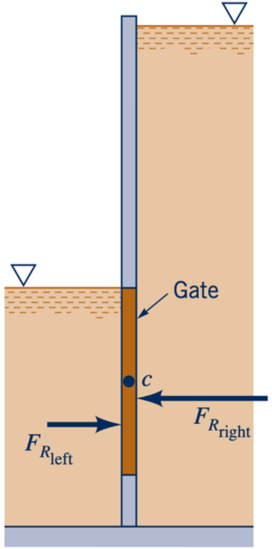
- If x’ and y’ are symmetry axes of the object, i.e., the submerged area is symmetrical with respect to an axis (like for a rectangle and a circle):

  $I_{xy_C} = 0 \Rightarrow \boldsymbol{x_R = x_C}$ → the application point lies on the same line as the centroid!

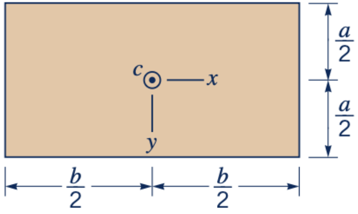
- In the y-direction, y_R is always lower than y_C because $\frac{I_{x_C}}{y_C A} > 0$ always!

- If y_C increases, the center of pressure y_R moves closer to the centroid!

- Since $y_C = h_C / sin\theta$, if h_C increases or θ decreases, then y_C increases so y_R move closer to the centroid!



Centroidal coordinates and moments of inertia for some common areas



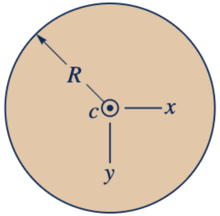
$A = ba$

$I_{xc} = \frac{1}{12} ba^3$

$I_{yc} = \frac{1}{12} ab^3$

$I_{xyc} = 0$

(*a*) Rectangle

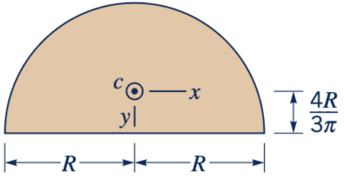


$A = \pi R^2$

$I_{xc} = I_{yc} = \frac{\pi R^4}{4}$

$I_{xyc} = 0$

(*b*) Circle



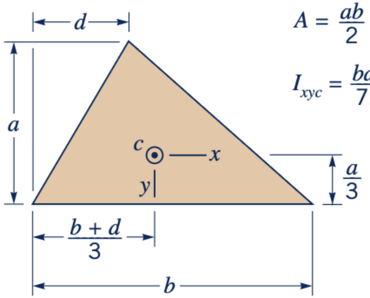
$A = \frac{\pi R^2}{2}$

$I_{xc} = 0.1098 R^4$

$I_{yc} = 0.3927 R^4$

$I_{xyc} = 0$

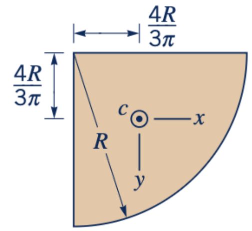
(*c*) Semicircle



$A = \frac{ab}{2}$ $\quad I_{xc} = \frac{ba^3}{36}$

$I_{xyc} = \frac{ba^2}{72}(b - 2d)$

(*d*) Triangle



$A = \frac{\pi R^2}{4}$

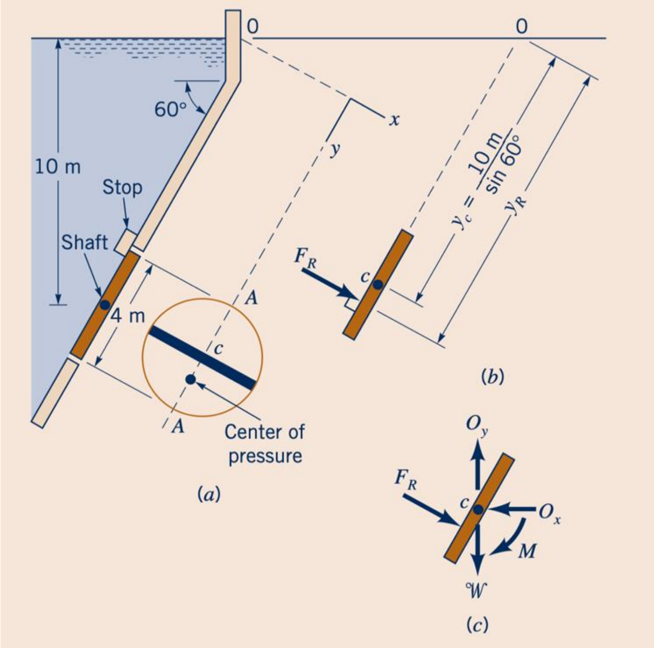
$I_{xc} = I_{yc} = 0.05488 R^4$

$I_{xyc} = -0.01647 R^4$

(*e*) Quarter circle

Hydrostatic forces on a plane surface

Example



Given: The 4-m-diameter circular gate in figure here is located in the inclined wall of a large reservoir containing water ($\gamma = 9.80\ kN/m^3$). The gate is mounted on a shaft along its horizontal diameter, and the water depth is 10 m above the shaft.

Determine:

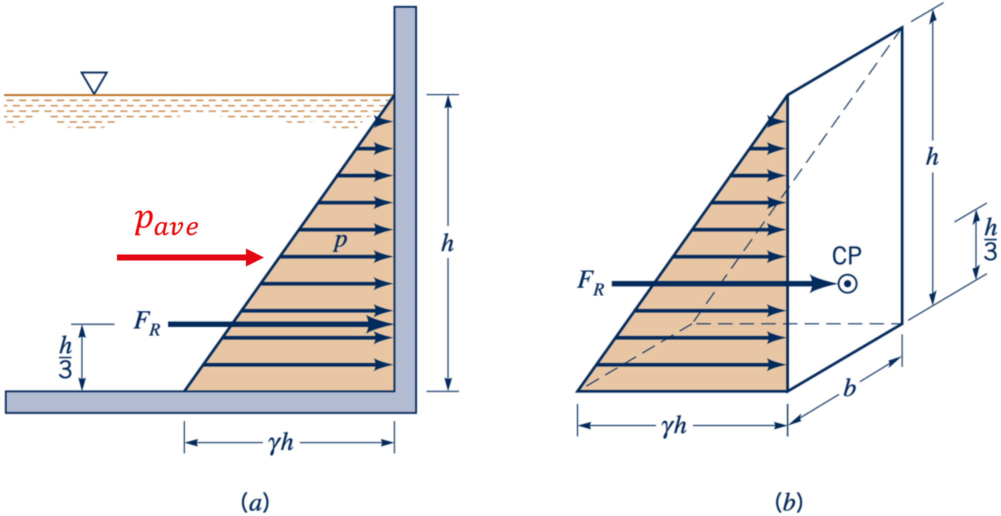
a) the magnitude and location of the resultant force exerted on the gate by the water and

b) the moment that would have to be applied to the shaft to open the gate.

Hydrostatic forces on a plane surface

Pressure prism

An informative and useful graphical interpretation can be made for the force developed by a fluid acting **on a plane rectangular area**. Consider the pressure distribution along a vertical wall of a tank of constant width b, which contains a liquid having a specific weight γ.



It is apparent from this diagram that the average pressure occurs at the depth $h/2$ and, therefore, the resultant force acting $A = bh$ on the rectangular area is:

$$F_R = p_{ave} * A = \gamma\left(\frac{h}{2}\right)A$$

(same result obtained earlier from generic case: force is specific weight * depth of centroid * area)

Hydrostatic forces on a plane surface

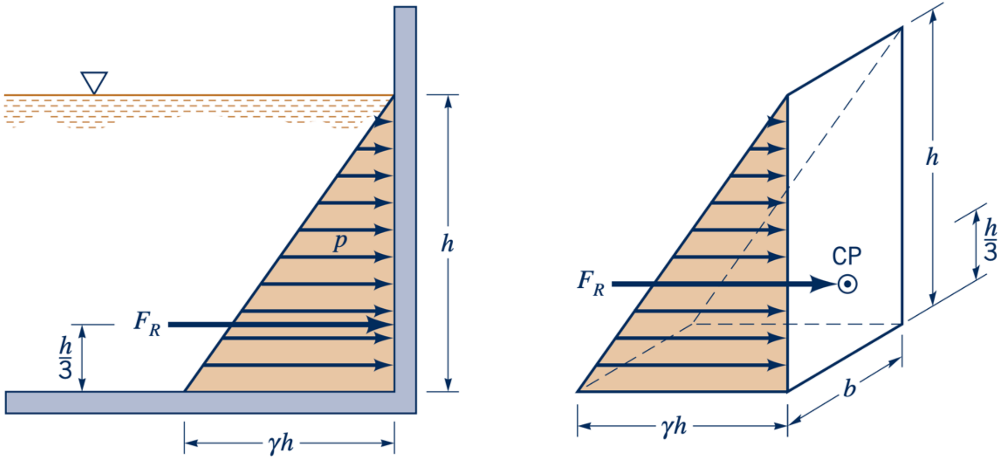
The force can also be seen as the ”volume” of the *pressure-prism* created by the triangular pressure distribution (the area) times the width of the wall:

$$F_R = volume = \left(\gamma h * \frac{h}{2}\right) * b = \gamma \left(\frac{h}{2}\right) A$$

Pressure “triangle” area

width

Same result



(*a*) (*b*)

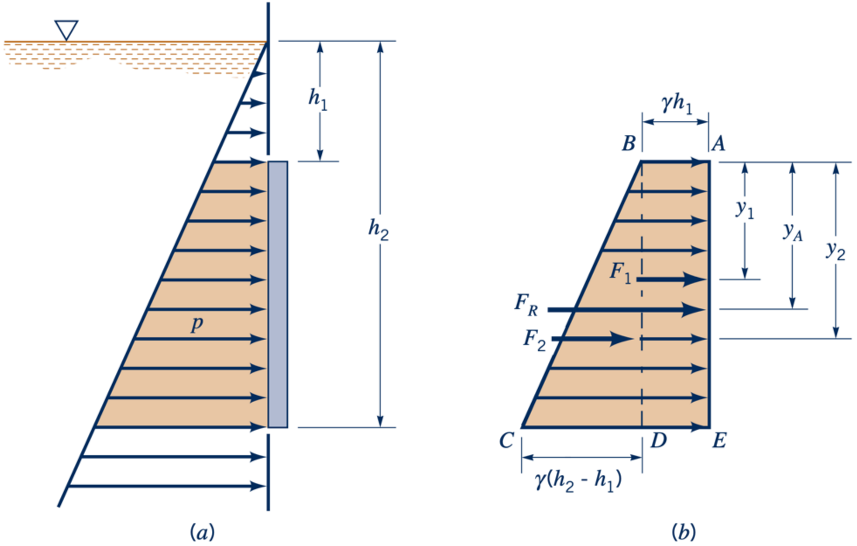
The resultant force must pass through the *centroid* of the pressure prism. For the volume under consideration the centroid is located along the vertical axis of symmetry of the surface and at a distance of $h/3$ above the base (since the centroid of a triangle is located at $h/3$ above its base).
This is consistent with the general method seen earlier.

Hydrostatic forces on a plane surface

This same graphical approach can be used for plane rectangular surfaces that do not extend up to the fluid surface.

In this instance, the cross section of the pressure prism is trapezoidal. The resultant force is still equal in magnitude to the volume of the pressure prism, and it passes through the centroid of the volume. Specific values can be obtained by decomposing the pressure prism into two parts, *ABDE* and *BCD.*

Therefore: $F_R = F_1 + F_2$



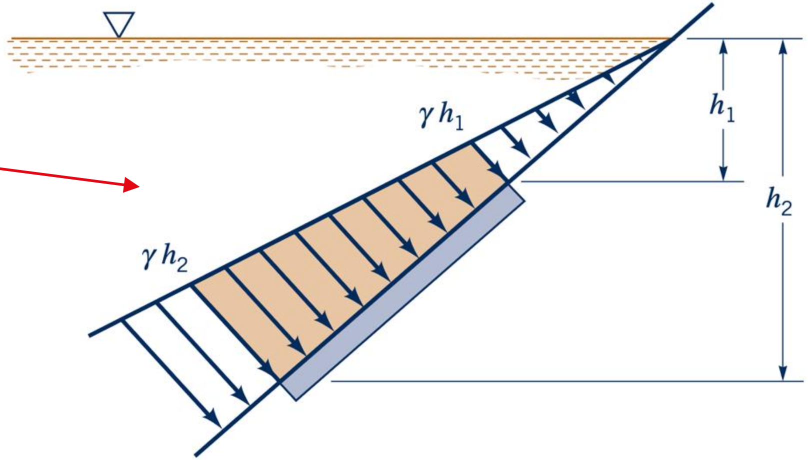
And the determined by summing moments about some convenient axis, such as one passing through *A:*

$$F_R y_A = F_1 y_1 + F_2 y_2$$

Hydrostatic forces on a plane surface

For inclined plane rectangular surfaces the pressure prism can still be developed, and the cross section of the prism will generally be trapezoidal.

Although it is usually convenient to measure distances along the inclined surface, the pressures developed depend on the vertical distances as illustrated.

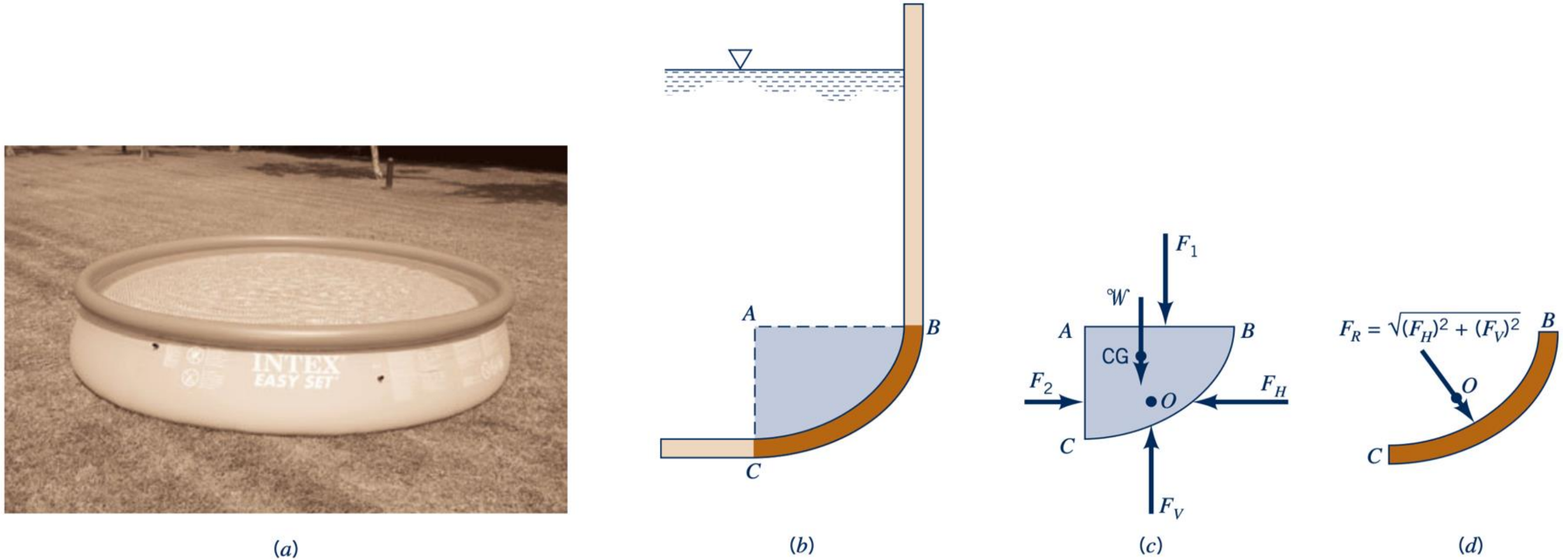


NOTE:

The use of pressure prisms for determining the force on submerged plane areas is convenient if the area is rectangular so the volume and centroid can be easily determined. However, for other nonrectangular shapes, integration would generally be needed to determine the volume and centroid. In these circumstances it is more convenient to use the equations developed earlier for generic shapes.

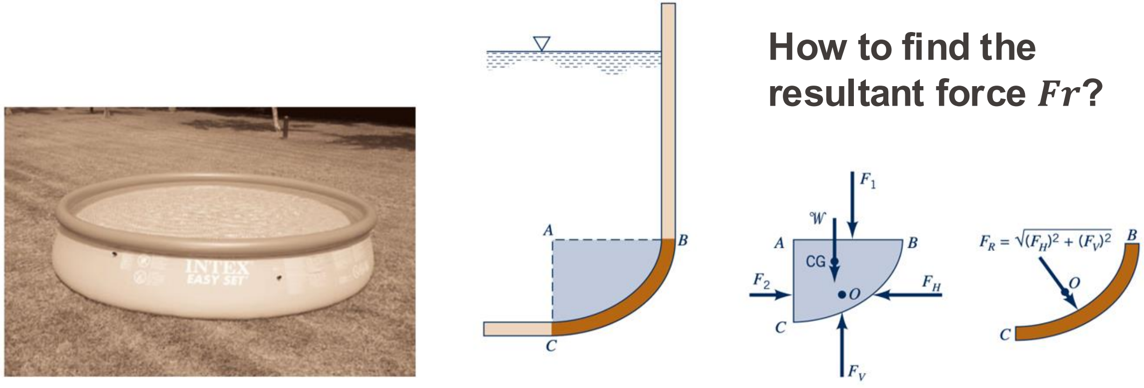
Hydrostatic forces on a curved surface

Immersed structures are often curved too, like pipes, portholes, the bottom of a pool, etc.
Integrating forces as we did before is rather tedious for curved surfaces, so an alternative method is usually used:



Resolve the EQUILIBRIUM of the volume of fluid enclosed by the curved surface and the horizontal and vertical projections of the surface

Hydrostatic forces on a curved surface



(a) (b) (c) (d)

How to find the resultant force Fr?

1) Isolate a volume of fluid bounded by the surface of interest (B-C) + horizontal plane A-B and vertical plane A-C

2) Resolve a free body diagram with all the forces which must be in equilibrium:

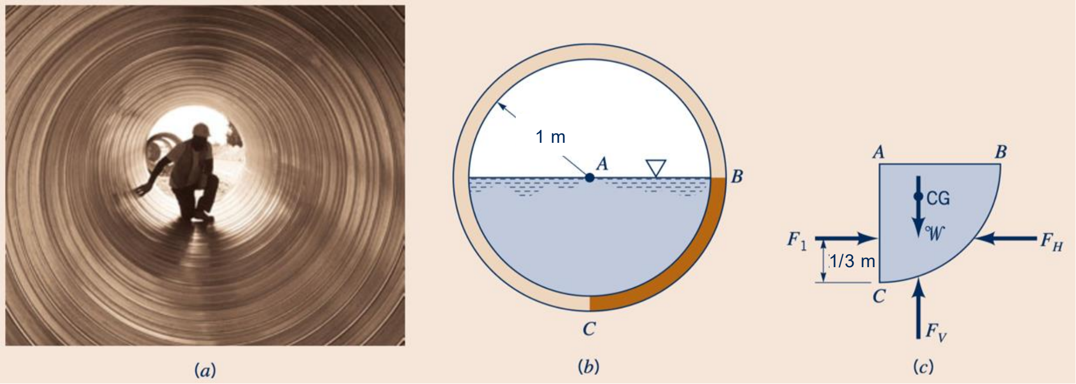
a) F_1 and F_2 are vertical and horizontal fluid forces on planar surfaces
b) W is the weight of the element and acts in the vertical direction
c) F_H and F_V are the horizontal and vertical components of the resultant force, i.e., the resistance that the wall does on the water
d) Balance the equilibrium:

$$\left.\begin{aligned} F_H &= F_2 \\ F_V &= F_1 + W \end{aligned}\right\} \quad F_R = \sqrt{F_H^2 + F_V^2}$$

F_R passes through the point O that can be located summing moment about an appropriate axis

Hydrostatic forces on a curved surface

Example



(a) (b) (c)

Given: A 2-m -diameter drainage conduit is half full of water at rest (see figure)

Determine:

Determine the magnitude and line of action of the resultant force that the water exerts on a 1 m length of the curved section *BC* of the conduit wall.