Hydraulic Engineering and Infrastructures

Civil Engineering Department

Uniform Flow

1 Flow in a straight channel †

In a straight channel reach, with width $B = 8\,\mathrm{m}$ and slope $S_0 = 1\,\%$, a discharge $Q = 5\,\mathrm{m}^3/\mathrm{s}$ flows under steady conditions (the discharge per unit width is therefore $q = 0.625\,\mathrm{m}^2/\mathrm{s}$). The roughness coefficient, expressed according to the Gauckler–Strickler[1] formula, is $n = \frac{1}{K_s} = 0.02\,\mathrm{s}/\mathrm{m}^{1/3}$.

Determine:

1. the characteristics of the uniform flow y_0, v_0, Fr_0, E_0;
2. the critical depth y_{CR} and the corresponding specific energy E_{CR}.

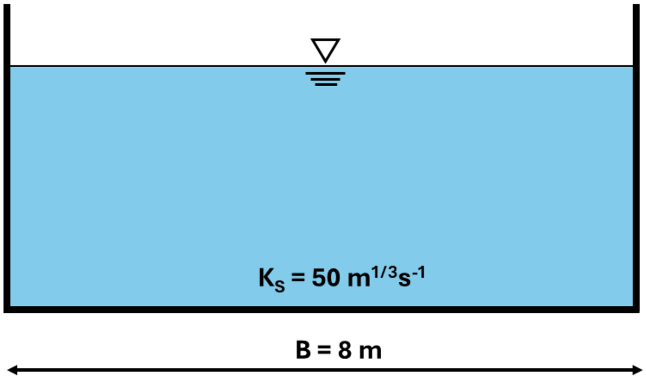

Figure 1: Sketch of a straight rectangular channel.

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[1] The Gauckler-Manning-Strickler (GMS) formula is also known as the Gauckler-Strickler formula, the Manning formula and the Manning-Strickler formula.

Objectives and guidance

The goal of this exercise is to understand how to determine the main hydraulic characteristics of a steady, uniform open-channel flow. This exercise helps to relate discharge, channel geometry, slope, and roughness to flow depth and velocity, and to distinguish between uniform and critical flow conditions.

Recommended procedure

1. Start by estimating the uniform flow depth using the given discharge, slope, and roughness, typically through an iterative approach.
2. Once the uniform depth is found, compute the corresponding flow velocity and identify the flow regime by evaluating the Froude number.
3. Determine the specific energy associated with the uniform flow conditions.
4. Calculate the critical depth and the corresponding specific energy based solely on the discharge per unit width.
5. Compare the uniform and critical conditions to assess whether the flow is subcritical or supercritical.

Solution

The determination of uniform flow characteristics is fundamental for solving most free-surface flow exercises. The first step consists of calculating the uniform flow depth y_0, which will then allow subsequent determination of other quantities.

This depth is calculated using a reformulation of the Strickler formula, in such a way as to make the depth explicit: an iterative expression is obtained which usually converges in at most 5 iterations:

$$y_0^{n+1} = \frac{nq}{\sqrt{S_0}} \left[\frac{B + 2y_0^n}{By_0^n} \right]^{2/3}$$

Note that the first term of the expression is constant and coincides with the uniform flow depth in the case of an infinitely wide channel ($Rh = y_0$). **The suggested first trial value**, which ensures rapid convergence both in subcritical and supercritical flow, is equal to the value of the uniform flow depth in an infinitely wide channel.

For a very wide channel, the hydraulic radius coincides with the flow depth ($R_h = y$), and the discharge per unit width can be expressed through the Gauckler–Strickler relation:

$$q = \frac{1}{n}\, y_\infty^{5/3} \sqrt{S_0}.$$

Solving for y gives:

$$y_\infty = \left(\frac{nq}{\sqrt{S_0}} \right)^{3/5}.$$

Substituting the known values $q = 0.625\ \mathrm{m}^2/\mathrm{s}$, $K_s = 50\ \mathrm{m}^{1/3}/\mathrm{s}$, and $S_0 = 0.01$:

$$y_\infty = \left(\frac{0.625}{50\sqrt{0.01}} \right)^{3/5} = (0.125)^{3/5} \approx 0.287\ \mathrm{m}.$$

Hence, the first trial value for the iterative computation is

$$y_0^{(0)} \approx 0.29\ \mathrm{m}.$$

Once we've iterated enough times to converge to a solution, the second step is to calculate the velocity v_0, expressed as the ratio between discharge per unit width and the uniform depth. The Froude number Fr_0 is then obtained, useful for distinguishing between slow and fast flows depending on whether its value is less than or greater than unity. Finally, the specific energy E_0 is computed:

$$v_0 = \frac{q}{y_0}, \quad Fr_0 = \frac{v_0}{\sqrt{g y_0}}, \quad E_0 = y_0 + \frac{v_0^2}{2g},$$

$$y_0 = 0.30\,\mathrm{m}, \quad v_0 = 2.1\,\mathrm{m/s}, \quad Fr_0 = 1.2, \quad H_0 = 0.52\,\mathrm{m},$$

The depth and specific energy under critical conditions, often very useful in solving more complex exercises, are functions only of the discharge per unit width:

$$y_{CR} = \left(\frac{q^2}{g}\right)^{1/3}, \quad E_{CR} = \tfrac{3}{2} y_{CR}$$

$$y_{CR} = 0.3415\,\mathrm{m}, \quad E_{CR} = 0.5123\,\mathrm{m}$$

2 Trapezoidal channel †

A trapezoidal channel has a base width $b = 5\,\mathrm{m}$, side slopes with an angle of 45°, a Manning roughness $n = 0.025\,\mathrm{s/m^{1/3}}$ and a slope $S = 0.17\,\%$. Its cross section is drawn in Figure 2.

1. Express $A(y)$, $P(y)$ and $R_H(y)$.
2. Compute the discharge if the normal depth is $y_n = 2\,\mathrm{m}$
3. Compute the normal depth if the discharge is $Q = 100\,\mathrm{m^3/s}$
4. Express the specific energy and find the depth for which it is minimized. What is this water depth, how is it called ? What is its value for $Q = 100\,\mathrm{m^3/s}$?
5. An engineer informs you that the flow speed should be above $2\,\mathrm{m/s}$ to ensure sediment transport, what is the minimal required water depth?.

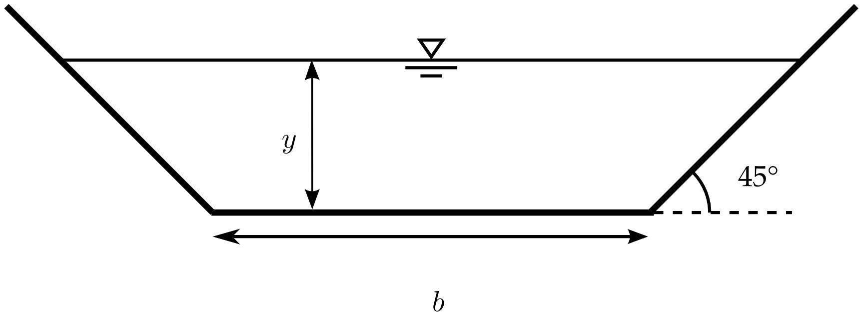

Figure 2: Cross-section of a trapezoidal channel

Objectives and guidance

This exercise focuses on expressing the normal depth and deriving the Froude number for a simple cross-section (and can easily be extended to any cross-section). It relies on a prismatic channel and a flow in normal conditions, showing that the computation of the discharge under such conditions is straight-forward while looking for flow depths is more challenging.

Recommended procedure

1. Express the geometric properties of the trapezoidal section: top width, wetted perimeter, cross-sectional area, and hydraulic radius as functions of the water depth.
2. For a known normal depth, compute the discharge using Manning's equation.

3. If the discharge is given, solve Manning's equation numerically (or iteratively) to find the normal depth.
4. Express the specific energy as a function of depth and discharge, then find the depth at which it is minimized. This defines the critical depth.
5. For a specified minimum flow velocity, set up the velocity–depth relation and solve numerically to determine the minimum water depth that satisfies the condition.

Solution

1. The top width $B(y)$ varies linearly with depth. with a slope $2/\tan(45°) = 2$:

   $$B(y) = b + 2y.$$

   The wetted perimeter follows a similar trend:

   $$P(y) = b + 2\sqrt{2}y.$$

   And the wetted area is:

   $$A(y) = \frac{1}{2}(B(y) + b) \cdot y = (b + y)y.$$

   Thus, the hydraulic radius is:

   $$R_h(y) = \frac{A(y)}{P(y)} = \frac{y(b + y)}{b + 2\sqrt{2}y}$$

2. Manning's law applies to normal conditions. Therefore, we can use it to estimate the discharge from the normal depth directly:

   $$\begin{aligned} Q &= \frac{1}{n}[A(y_n)/P(y_n)]^{2/3} A(y_n)\sqrt{S} && (1) \\ &= \frac{1}{n}\left[\frac{(b + y_n)y_n}{b + 2\sqrt{2}y_n}\right]^{2/3} (b + y_n)y_n\sqrt{S} && (2) \\ &= 27.7\,\mathrm{m}^3/\mathrm{s} && (3) \end{aligned}$$

3. Solving for y_n more complicated, we have to solve Equation 2 numerically (see the tutorial in the additional resources) to get

   $$y_n = 3.99\,\mathrm{m}.$$

4. The specific energy is $E = y + \frac{V^2}{2g}$. Rewriting the average speed in terms of discharge and wetted area:

   $$E = y + \frac{Q^2}{2gA(y)^2}.$$

   Since this function is convex, its minimum is found where its derivative is null. Let's first compute the derivative of the specific energy.

   $$\frac{\partial E}{\partial y} = 1 - \frac{Q^2}{gA(y)^3}\frac{\partial A}{\partial y} = 1 - \frac{Q^2 B(y)}{gA(y)^3},$$

Since $\partial A/\partial y$ is simply the top width B. This defines the Froude number.

$$\Rightarrow \frac{Q^2 B(y)}{gA(y)^3} = \left(\frac{V(y_c)}{\sqrt{gA(y_c)/B(y_c)}}\right)^2 = \mathrm{Fr}^2 = 1 - \frac{\partial E}{\partial y} \tag{4}$$

The minimum of the specific energy (found at $\partial E/\partial y = 0$) defines the critical depth y_c:

$$\mathrm{Fr} = \frac{V(y_c)}{\sqrt{gA(y_c)/B(y_c)}} = 1 \tag{5}$$

Applying it to the trapezoidal section, we get:

$$1 = \frac{Q^2(b + 2y_c)}{g(b + y_c)^3 y_c^3}$$

Whose solution is

$$y_c = 2.83\,\mathrm{m}.$$

5. This is the same problem as question 3 where the speed is imposed instead of the discharge. We seek to solve

$$\begin{aligned} V = \frac{Q}{A(y_n)} &= \frac{1}{n} R_H(y_n)^{2/3}\sqrt{S}, \\ &= \frac{1}{n}\left[\frac{(b + y_n)y_n}{b + 2\sqrt{2}y_n}\right]^{2/3}\sqrt{S}, \end{aligned}$$

Which is also an implicit equation. Its solution is

$$y = 2.04\,\mathrm{m}.$$

3 Simple step ††

A channel with a width of 1 m has a singularity, as shown in Figure 3. Draw an E vs y diagram with the different flow depths corresponding to the singularity in Figure 3. Assume that $Q = 5\ \mathrm{m}^3/\mathrm{s}$, that the step height is $a = 30\ \mathrm{cm}$, and that there is only one local energy loss of $h_s = 0.15\ \mathrm{m}$ between sections (2) and (3).

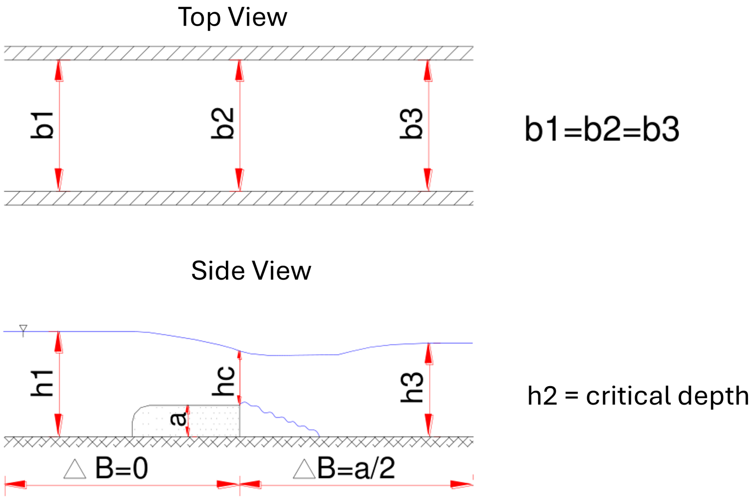

Figure 3: Step installed in a rectangular channel of constant width.

Objectives and guidance

This exercise focuses on understanding specific energy in open-channel flow, how a bed step modifies the flow regime, and how the energy equation with bed elevation changes and local energy losses can be used to determine the flow depths at different sections. Students should use the specific energy relationship to calculate the critical, upstream, and downstream flow depths, and represent the results on an E–y diagram.

Recommended procedure

1. Determine the critical depth from the discharge and channel geometry.
2. Apply the specific energy balance between sections (1) and (2), accounting for the step height, to find the upstream flow depth(s).
3. Apply the energy balance between sections (2) and (3), including the specified local loss h_s, to determine the downstream depth(s).

4. Plot the E–y diagram, the line $E = y$, and indicate the relevant points for each section.

Solution

To analyze the effect of a bed step on the flow in a rectangular channel, we use the concept of **specific energy** and apply the **energy equation** between different sections of the channel.

Given data:

- Channel width: $b = 1\ \mathrm{m}$
- Discharge: $Q = 5\ \mathrm{m}^3/\mathrm{s}$
- Step height: $a = 0.30\ \mathrm{m}$
- Local energy loss between sections (2) and (3): $h_s = 0.15\ \mathrm{m}$

The analysis involves four main steps:

1. Determine the critical depth.
2. Relate the upstream flow depth (section 1) to the critical section (section 2).
3. Relate the critical section to the downstream section (3), including the local loss.
4. Represent all the results on the E–h diagram.

1. Critical depth

The critical depth corresponds to the minimum specific energy for a given discharge. It is obtained by imposing the critical flow condition:

$$Fr^2 = 1 \quad \Longleftrightarrow \quad \frac{Q^2 b}{g\, A^3} = 1$$

For a rectangular channel, $A = b\, h$ and $b = 1\ \mathrm{m}$, so:

$$\frac{Q^2}{g\, b^2\, y_c^3} = 1 \quad \Longrightarrow \quad y_c = \sqrt[3]{\frac{Q^2}{g\, b^2}}$$

Substituting the numerical values:

$$y_c = \sqrt[3]{\frac{25}{9.8 \times 1^2}} = 1.366\ \mathrm{m}$$

This critical depth will occur just upstream of the steps (section 2), as the abrupt bed elevation change forces the flow to adjust to a critical state.

2. Upstream depth (section 1)

Between sections (1) and (2) there are no local losses. Applying the energy equation:

$$E_1 + z_1 = E_2 + z_2$$

Taking $z_1 = 0$ as reference and $z_2 = a = 0.30\ \mathrm{m}$, and knowing that section (2) is critical:

$$y_1 + \frac{Q^2}{2gb^2y_1^2} = \underbrace{y_c + \frac{Q^2}{2gb^2y_c^2}}_{\frac{3}{2}y_c} + a$$

Substituting:

$$y_1 + \frac{25}{2 \times 9.8\, y_1^2} = 1.366 + \frac{25}{2 \times 9.8 \times 1.366^2} + 0.3 = 2.349\,\mathrm{m}$$

Multiplying through by y_1^2:

$$y_1^3 - 2.349\, y_1^2 + 1.2755 = 0$$

This is a cubic equation, which typically has two positive real roots: one corresponding to supercritical flow (shallow depth, high velocity) and another to subcritical flow (deeper depth, lower velocity). Both solutions can be found iteratively with a first initial value below y_c and a second one above y_c.

The solutions are:

$$y_1 = \begin{cases} 0.957\ \mathrm{m} & \text{(supercritical / torrential)} \\ 2.044\ \mathrm{m} & \text{(subcritical / fluvial)} \\ -0.65\ \mathrm{m} & \text{(non-physical)} \end{cases}$$

Depending on the upstream boundary conditions, the flow may adopt either the supercritical or subcritical depth.

3. Downstream depth (section 3)

Between sections (2) and (3), a local energy loss of $h_s = 0.15\ \mathrm{m}$ must be included. The energy equation reads:

$$E_2 + z_2 = E_3 + z_3 + h_s$$

The downstream bed elevation is the same as at section 1, $z_3 = 0$. Therefore:

$$y_c + \frac{Q^2}{2gb^2y_c^2} + a = y_3 + \frac{Q^2}{2gb^2y_3^2} + h_s$$

Substituting numerical values:

$$1.366 + \frac{25}{2 \times 9.8 \times 1.366^2} + 0.3 = y_3 + \frac{25}{2 \times 9.8\, y_3^2} + 0.15$$

Solving this equation gives:

$$y_3 = \begin{cases} 1.056\ \mathrm{m} & \text{(supercritical / torrential)} \\ 1.810\ \mathrm{m} & \text{(subcritical / fluvial)} \\ -0.667\ \mathrm{m} & \text{(non-physical)} \end{cases}$$

Again, two possible flow regimes are possible downstream of the step, depending on the hydraulic control conditions.

4. E–y diagram

The specific energy values for each section are:

$$E_1 = y_1 + \frac{Q^2}{2gb^2y_1^2} = 2.349\ \mathrm{m}$$

$$E_2 = 2.049\ \mathrm{m}$$

$$E_3 = 2.199\ \mathrm{m}$$

Plotting these points on the E–y diagram allows us to clearly identify the two possible depths (supercritical and subcritical) for each specific energy level, as well as the critical depth.

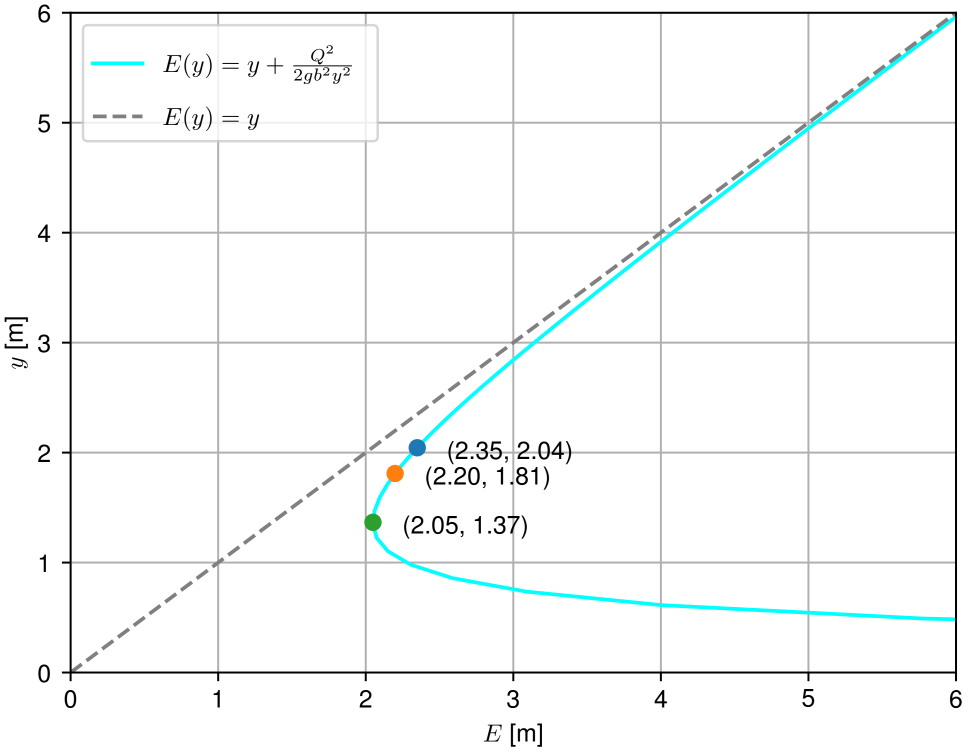

Figure 4: Specific energy vs. flow depth diagram. The cyan curve represents the energy–depth relation, the dashed black line corresponds to $E = y$, and the red points indicate the computed solutions at the upstream, critical, and downstream sections.

This diagram is a powerful tool to visualize how the bed step and energy losses affect the flow regime, and why multiple depths are possible for the same discharge.