Hydraulic Engineering and Infrastructures

Civil Engineering Department

Fluid Mechanics Review

1 Pressure of water flowing inside a pipe †

A U-tube piezometer is used to measure the pressure of water flowing inside a pipe. The fluid used in the piezometer is mercury, which has a specific weight $\gamma_{Hg} = 133000$ Nm^{-3}. Two configurations are shown in Fig. 1, the first on the left and the second on the right.

Calculate:

(a) the gauge pressure P_G of the water measured by the manometer.

(b) the new heights h_1' and h_2' if the pressure inside the pipe is $P_G' = P_G - 30$ kPa.

Solution

The U-pipe piezometer consists of two vertical transparent tubes that communicate with each other. One of the tubes also communicates with the atmosphere, while the other one communicates with the measurement environment.

Recalling the **hydrostatic law** $p + \gamma_h = const$, we can see that if two points belong to the same fluid while being at the same depth, then they are subjected to the same pressure. Then, any horizontal surface contained within the same fluid is an isobaric surface.

The horizontal surface passing through the water-mercury interface (BB' in the first case and CC' in the second) is an isobaric surface. Conveniently, point B (and point C), can be considered as belonging to the water or to the mercury fluids indistinctly.

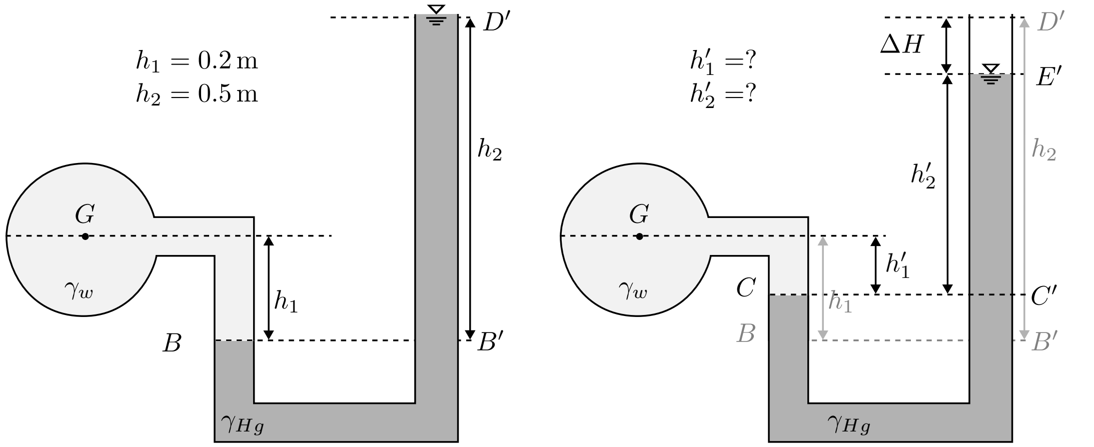

Figure 1: Two configurations of the U-tube piezometer.

For the first configuration, using the isobar definition applied to the BB' surface, we have $P_B = P_{B'}$. Using the hydrostatic law for mercury, $P_{B'} = P_{D'} + (\gamma_{Hg} \times h_2)$. As $P_{D'}$ is the reference atmospheric pressure, it is null and we get: $P_{B'} = \gamma_{Hg} \times h_2$.

The pressure at point B can be expressed with respect to the centroid of the conduit, G: $P_B = P_G + (\gamma_W \times h_1)$.

Then,

$$\begin{aligned} P_B &= P_{B'} \\ P_G + (\gamma_W \times h_1) &= \gamma_{Hg} \times h_2 \\ P_G &= \gamma_{Hg} \times h_2 - \gamma_W \times h_1 \end{aligned}$$

Which gives:

$$P_G = 64.54 \text{ kPa}$$

In the second configuration, we apply the same procedure on the isobaric surface CC', noting that the free surface is at point E'.

The new mercury height above the isobaric surface is $h'_2 = h_2 - 2\Delta H$. The new water height is $h'_1 = h_1 - \Delta H$.

Applying the hydrostatic law on both sides, we get:

$$P_{C'} = \gamma_{Hg} \times h_2' = \gamma_{Hg} \times (h_2 - 2\Delta H)$$
$$P_C = P_{G'} + \gamma_W \times h_1' = P_{G'} + \gamma_W \times (h_1 - \Delta H)$$

Since $P_C = P_{C'}$, we have:

$$P_{G'} + \gamma_W \times (h_1 - \Delta H) = \gamma_{Hg} \times (h_2 - 2\Delta H)$$

After expressing ΔH, we can obtain h_1' and h_2':

$$h_1' = 8.291 \times 10^{-2} \text{ m}$$
$$h_2' = 2.658 \times 10^{-1} \text{ m}$$

2 Concomitant chambers †

A tank consists of two independent chambers connected by a square opening of side 0.4 m, which is kept closed by a gate of the same shape as the opening and hinged at the point O. The left chamber is sealed and contains pressurized gas, while the right chamber contains water up to a height y above the hinge. The left chamber is connected to a piezometer containing the gas with a column of water of height $h = 1.2$ m above.

Calculate:

(a) the pressure in the left chamber P_{GAS}.

(b) the level y of water in the right tank for which the conditions shown in Fig. 2 give equilibrium for the gate.

(c) sketch all the forces acting on the gate, specifying magnitude, direction, and sense.

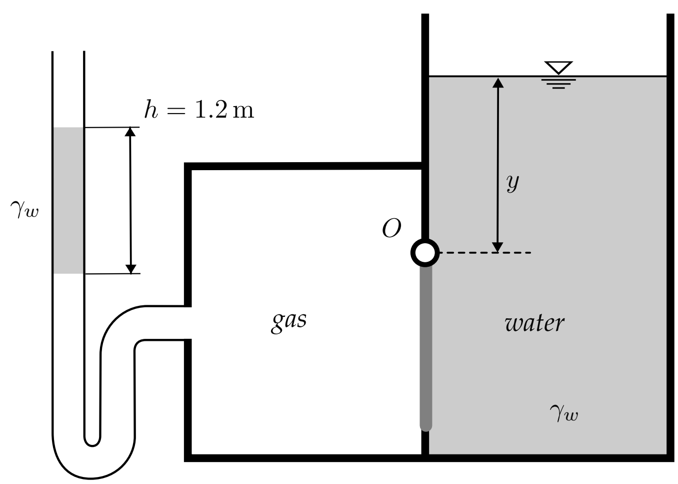

Figure 2: Problem configuration.

Solution

The height of the fluid inside the piezometer allows the pressure of the gas in the left chamber to be calculated using the hydrostatic equation: $P_{GAS} = P_{atm} + \gamma_W \times h$. As P_{atm} is the reference atmospheric pressure, we set it to 0 Pa.

$$P_{GAS} = 11.78 \text{ kPa}$$

Knowing this value, the height of the water above the hinge, y, is found by solving the moment-equilibrium equation about point O. in which the only unknown is the water level y.

$$y = 0.93 \text{ m}$$

3 Force on a cylindrical aquarium † † †

A tank filled with a fluid ($\gamma_\text{f} = \gamma_\text{water}/2 = 4900\,\text{N/m}^3$) and water ($\gamma_\text{water} = 9800\,\text{N/m}^3$) has a cylindrical aquarium of width $w = 10\,\text{m}$ and radius $r = 3\,\text{m}$ placed right at the interface between the two fluids as shown in Figure 3. For a height $H = 10\,\text{m}$, compute the force applied on the aquarium and its application point on the aquarium's surface.

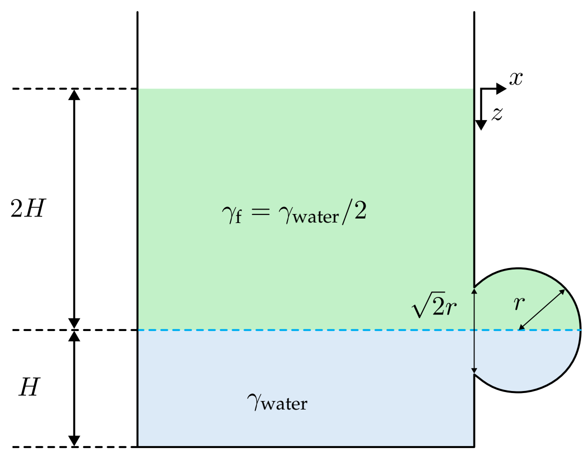

Figure 3: A cylindrical aquarium with its bottom half filled with water and the top half with a light fluid.

Solution † † †

To compute the pressure force F within the aquarium, we integrate it on its surface:

$$\mathbf{F} = \int dF = \int -p \cdot \hat{n} dS \text{ with } \hat{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

The pressure inside the aquarium depends on the depth z at which we are. In the aquarium, this means that it depends on which angle θ we are at.

$$p(z) = \begin{cases} \gamma_t z & \text{if } z < 2H \\ 2H\gamma_t + \gamma_{\text{water}}(z - 2H) & \text{if } z > 2H \end{cases}$$

$$p(\theta) = \begin{cases} \gamma_t(2H - r\sin\theta) & \text{if } 0 < \theta < 3\pi/4 \\ 2H\gamma_t - \gamma_{\text{water}} r \sin\theta & \text{if } -3\pi/4 < \theta < 0 \end{cases}$$

As $\gamma_t = \gamma_{\text{water}}/2$, we obtain:

$$\mathbf{F} = \begin{pmatrix} F_H \\ F_V \end{pmatrix} = \begin{pmatrix} \gamma_{\text{water}} r(H\sqrt{2} + \frac{1}{8}r)w \\ \gamma_{\text{water}} r^2(\frac{3}{8} + \frac{9\pi}{16})w \end{pmatrix} = \begin{pmatrix} 4.27\,\text{MN} \\ 1.89\,\text{MN} \end{pmatrix}$$

To determine the point of application of the force (x', y'), we can use the following formulas:

$$\begin{aligned} F_V \cdot x' &= \int x \, dF_V = \int_{CS} x \cdot p(z)\hat{n}_y \cdot dA \\ &= \int_{-\frac{3\pi}{4}}^{0} r(1 + \cos\theta)(2H\gamma_t - \gamma_{\text{water}} r \sin\theta)\sin(\theta) wr \, d\theta \\ &\quad + \int_{0}^{\frac{3\pi}{4}} r(1 + \cos\theta)(2H - r\sin\theta)\gamma_t \sin(\theta) wr \, d\theta \end{aligned}$$

$$\Rightarrow x' = \left(1 + \frac{2\sqrt{2}}{6 + 9\pi}\right) r = 3.25\,\text{m}$$

The application point of the horizontal forces can be computed with the projected force in the x-direction.

$$\begin{aligned} F_H \cdot z' &= \int z \, dF_H = \int_{CS} z \cdot p(z)\hat{n}_x \cdot dA \\ &= \int_{-3\pi/4}^{3\pi/4} z \cdot p(z)\hat{n}_x \cdot wr \, d\theta \\ &= \int_{2H-\sqrt{2}r/2}^{2H+\sqrt{2}r/2} z \cdot p(z) \cdot dA \quad \text{(projected area)} \end{aligned}$$

$$\Rightarrow z' = 2H + \frac{\sqrt{2}r^3 w \gamma_{\text{water}}}{8F_H} = 20.11\,\text{m}.$$

It can be noted that the action line of the force passes through the center of the circle in Figure 4 since the appliction point and the center of the circle are aligned with an angle α with respect to the horizontal.

$$\beta = \arctan \frac{y' - 2H}{x' - r} = \arctan \frac{F_y}{F_h} = \alpha = 23.9°$$

.

This is because all infinitesimal forces $p(z) \cdot \hat{n} \, dA$ are aligned with the radii. Computing the torque around the center of the circle would yield

$$M = \int (r\hat{n}) \wedge p(z)\hat{n} \, dS = \int r \begin{pmatrix} \cos\theta \\ \sin\alpha \end{pmatrix} \wedge \begin{pmatrix} \cos\theta \\ \sin\alpha \end{pmatrix} p(z) \, dS = \int 0 \, dA = 0.$$

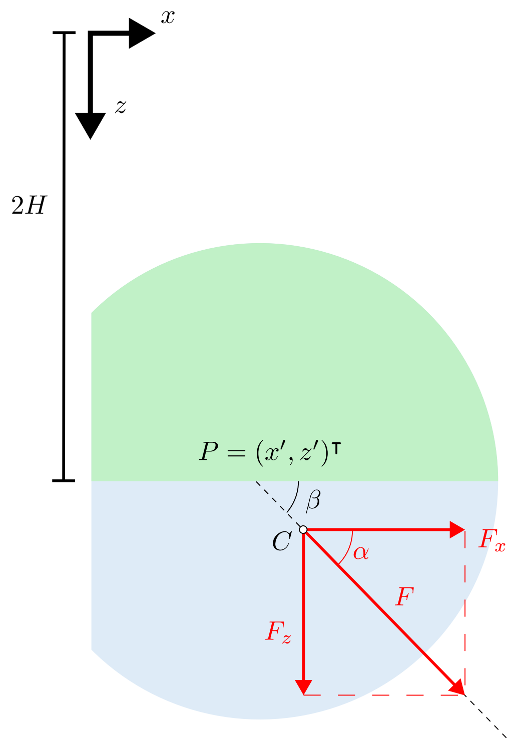

Figure 4: Point of application of the force. As long as the force is left along its line of action, its torque is the same.

The crossing between the force's action line and the circle is finally

$$P_{\text{on circle}} = \begin{pmatrix} r \\ 2H \end{pmatrix} + \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} = \begin{pmatrix} r \\ 2H \end{pmatrix} + \begin{pmatrix} 2.74\,\text{m} \\ 1.21\,\text{m} \end{pmatrix} = \begin{pmatrix} 5.74\,\text{m} \\ 21.2\,\text{m} \end{pmatrix}.$$

Alternative solution ††

The force applied on aquarium can be computed with a geometrical method : instead of computing the integral over the control surface, directly compute the weight of the weight of the column of water above the surface. The area of the different columns is drawn on Figure 5.

The upwards force is equal to the weight of the the area A_1:

$$F_{y1} = \gamma_1 \cdot A_1 = \gamma_1 \cdot \left(\frac{\sqrt{2}}{2} + 1 \right) r \cdot 2H - \gamma_1 \cdot A_2 \tag{1}$$

$$= \gamma_1 \cdot Hr \left(\sqrt{2} + 2 \right) - \gamma_1 \cdot \left(\frac{3\pi}{8} + \frac{1}{4} \right) r^2. \tag{2}$$

Similarly, the weight of the upwards-pushing column is :

$$F_{y2} = \gamma_1 \cdot A_1 + \gamma_1 A_2 + \gamma_2 A_3 = \gamma_1 \cdot \left(\frac{\sqrt{2}}{2} + 1 \right) r \cdot 2H + \gamma_2 \cdot A_2 \tag{3}$$

$$= \gamma_1 \cdot Hr \left(\sqrt{2} + 2 \right) + \gamma_2 \cdot \left(\frac{3\pi}{8} + \frac{1}{4} \right) r^2. \tag{4}$$

So the vertical force on the aquarium is

$$F_y = -W_1 + W_2 = (\gamma_1 + \gamma_2) \cdot \left(\frac{3\pi}{8} + \frac{1}{4} \right) r^2$$

Note that for $\gamma_1 = \gamma_2$, the result respects Archimedes' principle. For $\gamma_1 = \gamma_2/2$,

$$F_y = \frac{3}{2}\gamma_2 \cdot \left(\frac{3\pi}{8} + \frac{1}{4} \right) r^2 = 1.89\,\mathrm{MN}.$$

The horizontal force can be computed with the projected area for the top and bottom half separately:

$$F_{x1} = \frac{\sqrt{2}}{2} r \gamma_1 \cdot \left(2H - \frac{\sqrt{2}/2}{2} r \right) \tag{5}$$

$$F_{x2} = \frac{\sqrt{2}}{2} r \cdot \left(\gamma_1 \cdot 2H + \gamma_2 \cdot \frac{\sqrt{2}/2}{2} r \right) \tag{6}$$

$$\Rightarrow F_x = \sqrt{2} r \gamma_2 H + (\gamma_2 - \gamma_1) \frac{r^2}{4} \tag{7}$$

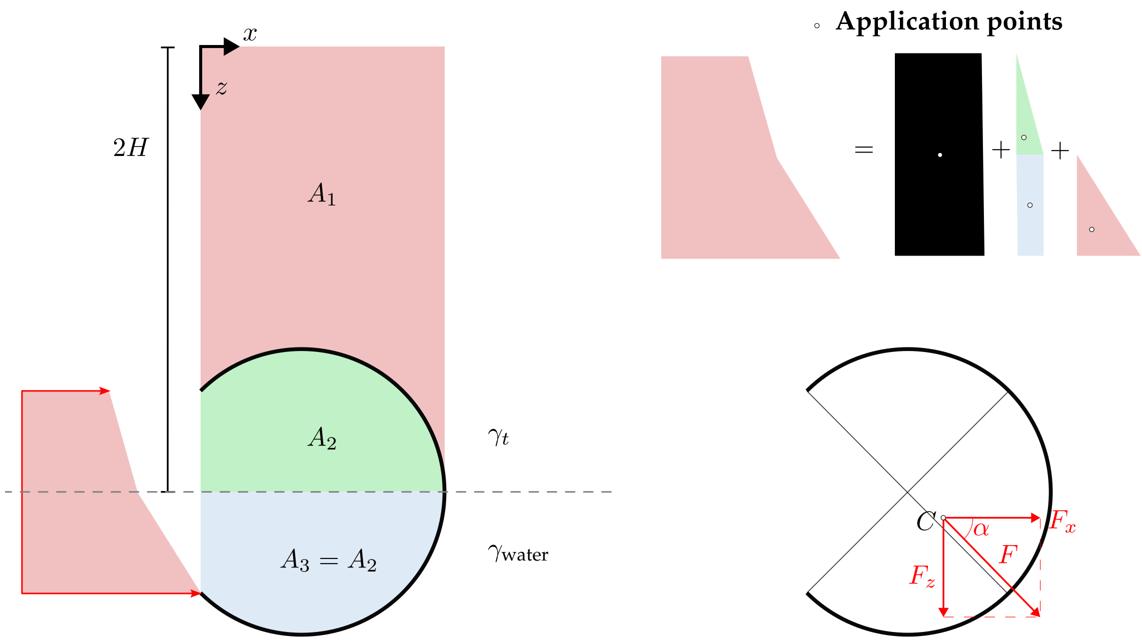

Figure 5: Computing the vertical force on the aquarium with a geometric method.

Second alternative solution †

We can make use of the tabulated moments of inertia and the parallel axis theorem to avoid using the integrals. Since the values for rectangular shapes are tabulated (see course notes), we can use them to compute the forces and the application points. Note that we are making use of the projected area. A sketch of the solution is given in Figure 6

For the top half, the center of gravity is $z_{C1} = 2H - h/2 = 18.94\,\mathrm{m}$, its area is $A_1 = wh = 21.21\,\mathrm{m}^2$ and the application point is

$$z_{R1} = z_{C1} + \frac{I_{yC1}}{z_{C1}A_1} = 18.96\,\mathrm{m}.$$

And the force is $F_1 = p(z_{C1})A_1 = \gamma_f z_{C1} wh$.

For the bottom half, the change in density pushes us to define a new origin $z' = z - H$ to find a hydrostatic pressure distribution (with slope γ_{water}) matching that of the bottom half for the parallel axis theorem to apply.

Having defined this new origin, the center of mass $z'_{C2} = H + h/2 = z_{C2} - H = 11.06\,\mathrm{m}$, the application point is $z_{R2} = z'_{R2} + H = \frac{I_{yC2}}{z'_{C2}A_2} + H = 21.09\,\mathrm{m}$ and the force is $F_1 = \gamma_{\text{water}} \cdot (H + h/2)A_2$.

The application point of the total force is found with

$$z_R = \frac{F_1 z_{R1} + F_2 z_{R2}}{F_1 + F_2} = 20.11\,\mathrm{m}.$$

Which is in agreement with the previous results.

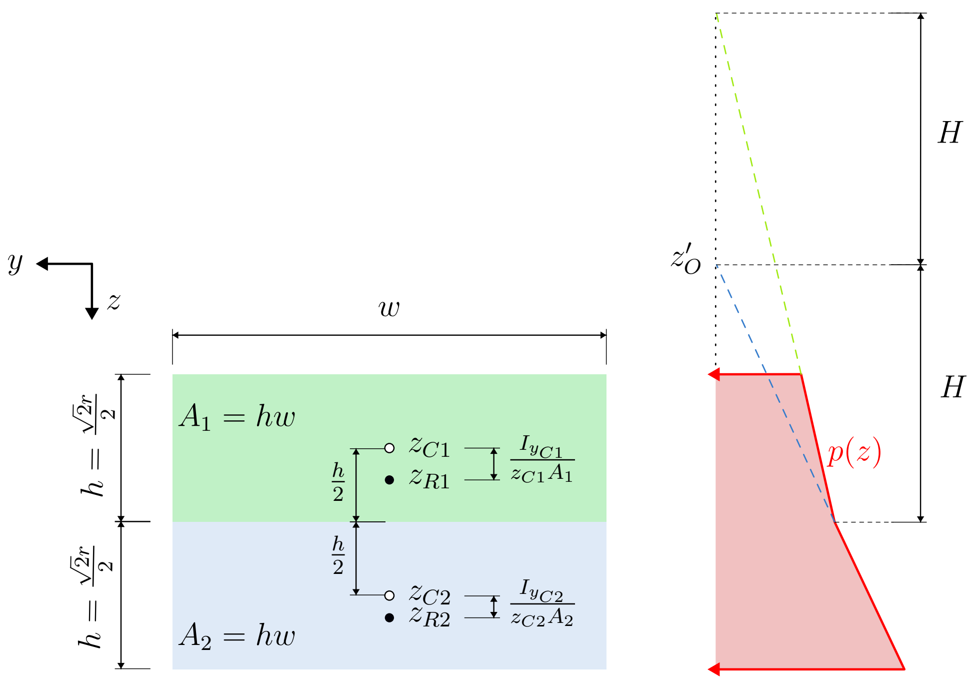

Figure 6: Drawing of the opening, its areas and application points.