Hydraulic Engineering and Infrastructures

Civil Engineering Department

Flow profiles

1 Hydraulic jump at slope change

A long spillway (slope of 10 % and Manning coefficient 0.025 $s/m^{1/3}$) discharges into a flat stilling basin. There, the water depth is controlled by a step before the water flows into a steep river. A small step height will change the flow lightly, but a larger step will force a subcritical flow further upstream of it and generate a hydraulic jump in the stilling basin. The channel is reactangular with a width of 2 m, has a discharge of 50 m^3/s and the stilling basin is 40 m long. What is the maximum step height before the hydraulic jump affects the spillway?

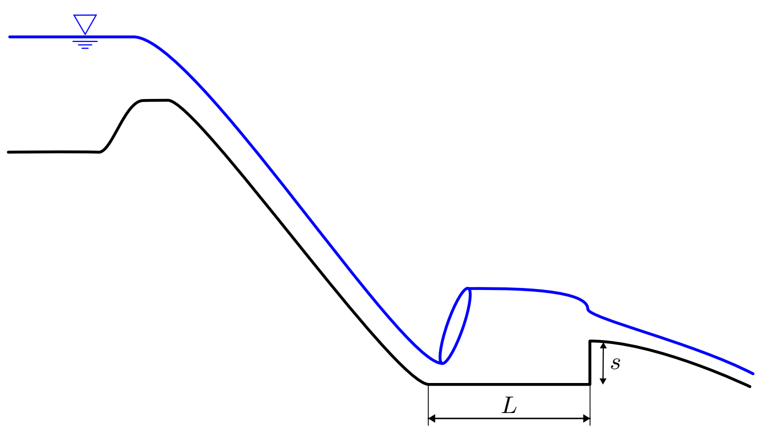

Figure 1: Schema of the problem

Objectives and guidance:

The aim of this exercise it to understand the concept of a conjugate height and use it to find the position of a hydraulic jump using the direct-step method.

Hints :

- For the spillway to be free, the hydraulic jump has to happen at the slope change or further downstream. Thus, there is a condition at the slope change: the depth directly downstream is the conjugate of the supercritical depth directly upstream.
- The direct step method can be computed both ways (upstream $\leftrightarrow$ downstream) independently of flow conditions.
- The depth of the step is defined by the jump in energy around the step. Consider that there is no energy loss at the step: $E(x = L - \varepsilon) = s + E(x = L + \varepsilon)$ with ε being a small distance.

Solution

The first step is to compute the critical depth in the stilling basin and the normal depth in the spillway. The assumption of normal flow in the spillway can be made if it is long enough to attain normal conditions.

$$y_{c,\text{basin}} = \left(\frac{Q}{\sqrt{g}b}\right)^{2/3} = 3.99\,\text{m}$$

$$y_{n,\text{spillway}} = \frac{nQ}{R_H^{2/3} b\sqrt{i}} \approx \frac{nQ}{y_{n,\text{spillway}}^{2/3} b\sqrt{i}} \Rightarrow y_{n,\text{spillway}} = 2.48\,\text{m}$$

As the depth after the hydraulic jump should be the conjugate depth of $y_{n,\text{spillway}}$, let it be

$$y_{\text{conjugate}} = \frac{y_{n,\text{spillway}}}{2}\left(\sqrt{1+8F_r^2} - 1\right) = 6.04\,\text{m}.$$

The latter is the depth we'll start from at the upstream end of the stilling basin when computing the profile. As the hydraulic jump is set to be at the end of the spillway, we are now looking to find the depth at the position $x = 40\,\text{m}$. It is an M2 profile. First of all, we need to express the Froude number, the hydraulic radius and the friction slope.

The Froude number, the hydraulic radius and the friction slope for a rectangular channel are expressed as follows :

$$F_r = \frac{Q}{\sqrt{gy}by}$$

$$R_H = \frac{by}{b+2y}$$

$$S_f = \frac{n^2V^2}{R_H^{4/3}}$$

The derivative dx/dy comes from the energy balance :

$$\frac{\Delta x}{\Delta y} = \frac{1-F_r^2}{S_0 - S_f}$$

The increments Δx are negative since the computation goes from downstream to upstream. The results are shown in Table 1 and Figure 2.

Once the profile is solved and the depth at $x = 40\,\mathrm{m}$ is interpolated, we can infer the necessary step height by assuming an energy balance between the upstream of the step $E_1 = E(x = 40\,\mathrm{m})$ and the critical flow over the step E_2.

$$E_1 = y_1 + \frac{Q^2}{2gb^2y_1^2} = \frac{3}{2}y_c + s = E_2 + s$$

$$\Rightarrow s = y_1 + \frac{Q^2}{2gb^2y_1^2} - \frac{3}{2}h_c = 0.294\,\mathrm{m}$$

Where y_1 was found at $x = 40\,\mathrm{m}$ by computing the hydraulic grade line from the conjugate depth $y_{\text{conjugate}} = 6.04\,\mathrm{m}$ of the normal depth in the spillway $y_n = 2.48\,\mathrm{m}$ at $x = 0\,\mathrm{m}$.

Table 1: Direct-step method to compute the profile in the stilling basin. The step Δy was chosen to attain critical depth in 5 iterations (4 increments): $\Delta y = (y_{\text{conjugate}} - y_{\text{critical}})/4$.

x	dy	y	V	F_r	R_H	S_f	$\Delta x/\Delta y$	$\overline{\Delta x/\Delta y}$	Δx
0.00	-0.51	6.04	4.14	0.54	0.86	0.01	-54.08	-46.55	23.80
23.80	-0.51	5.53	4.52	0.61	0.85	0.02	-39.02	-32.03	16.37
40.17	-0.51	5.02	4.98	0.71	0.83	0.02	-25.04	-18.55	9.48
49.65	-0.51	4.51	5.55	0.83	0.82	0.03	-12.06	-6.03	3.08
52.74		3.99	6.26	1.00	0.80	0.03	0.00		

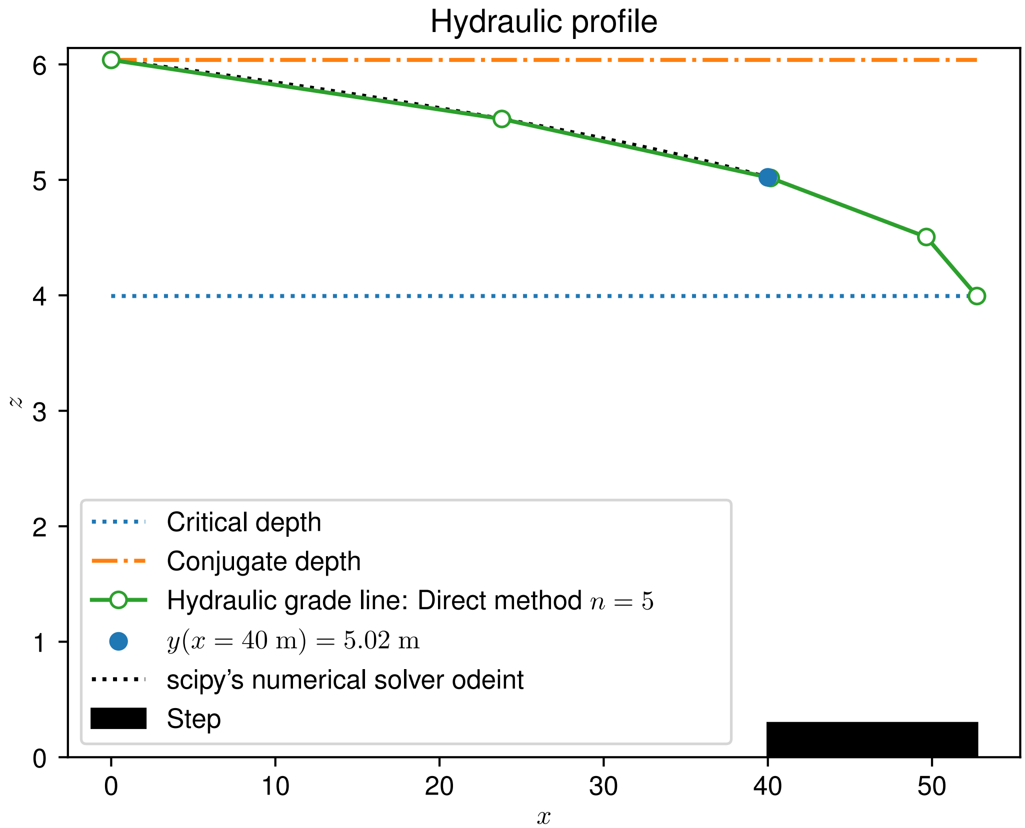

Figure 2: Hydraulic profile of the stilling basin.

2 Hydraulic jump with gate and weir

A group of students is setting up a laboratory experiment to reproduce the formation of a hydraulic jump. They use a flume having a width $b = 0.15$ m, a slope $S_0 = 0.001$, and a Manning coefficient $n = 0.0092$ m$^{-1/3}$s (Plexiglass). In the setup, they place an upstream gate with a constant water level and a constant discharge $Q = 0.011$ m^{3}s^{-1}. The gate vertical opening is $h_{gate} = 0.051$ m, and the minimum water depth reached downstream of the gate (vena contracta) is $y_2 = 0.025$ m. 15 meters downstream of the gate, they also place a $d_{weir} = 0.03$ m long-crested weir where choking occurs. Assume there are no losses at the transition on the weir.

Determine the location of the hydraulic jump x_{jump} and draw the resulting hydraulic profile.

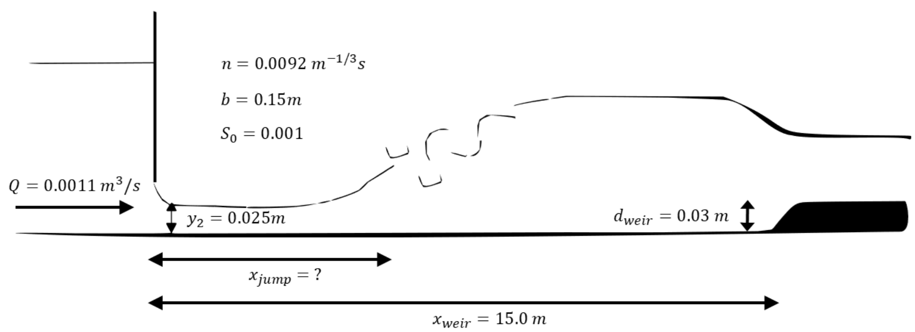

Figure 3: Setup of the flume experiment and expected outcome.

Solution

To solve this problem, you should approach it from the two sides. First, compute and draw the profile from the gate opening to downstream, pretending that the long-crested weir is not present. Consequently, compute and draw the conjugate depths that would result in the presence of a hydraulic jump. Similarly, compute and draw the water profile from the long-crested weir to upstream until the gate, assuming that the gate does not affect the flow. To compute these profiles you should use the middle point method for rectangular channels.

Once the three profiles (the one resulting from the gate, its conjugate, and the one resulting from the long-crested weir) are computed and drawn, the location of the hydraulic jump is found on the interpolation between the conjugate and the long-crested weir profiles.

Profile resulting from the gate opening

Before computing the whole profile we should determine the critical and normal flow depths. For that, we first compute the flow velocity at the vena contracta point (minimum depth downstream of the gate y_2, which is the initial point for the computation and drawing of the profile).

Reversing the discharge formula we find

$$v_2 = \frac{Q}{A_2} = \frac{Q}{y_2 b} = \frac{0.011\ \mathrm{m^3/s}}{0.025\ \mathrm{m} \cdot 0.15\ \mathrm{m}} = 2.87\ \mathrm{m/s}$$

and consequently

$$Fr_2 = \frac{v_2}{\sqrt{g y_2}} = \frac{2.87\ \mathrm{m/s}}{\sqrt{9.81\ \mathrm{m^2/s} \cdot 0.025\ \mathrm{m}}} = 5.8$$

Therefore, since $Fr > 1$, the flow is supercritical.

We compute the critical depth as

$$y_c = \left(\frac{Q^2}{b^2 g}\right)^{1/3} = \left(\frac{0.011^2\ \mathrm{m^3/s}}{0.15^2\ \mathrm{m} \cdot 9.81\ \mathrm{m^2/s}}\right)^{1/3} = 0.082\ \mathrm{m}$$

and the normal depth as

$$y_n = \left(\frac{nQ}{b{S_0}^{1/2}}\right)^{3/5} = \left(\frac{0.0092\ \mathrm{m^{-1/3}s} \cdot 0.011\ \mathrm{m^3/s}}{b{S_0}^{1/2}}\right)^{3/5}$$

Consider that for applying this equation we assume that the channel is infinitely large. The result is conservative compared to the equation for real rivers, for which you should solve the equation $nQ - \frac{(b \cdot y_n)^(5/3)}{(b+2y_n)^(2/3)} \cdot \sqrt{S_0} = 0$. Doing in for this exercise returns a value of approximately $y_n = 0.156$ m.

The water depth at the gate outlet is lower than the critical depth, and that confirms that the flow is supercritical. For this reason, when computing the whole profile you should stop at the critical depth. Continuing the computation to the normal depth would be wrong, because when the flow reaches the critical depth, the hydraulic jump occurs.

To compute the profile we use the middle point method. For that, we subdivide the profile in smaller sections. The initial point of the profile is $y_2 = 0.025$ m. The last point of the profile is the critical depth $y_c = 0.082$ m. Hence, the total height variation is $\Delta y = y_c - y_2 = 0.082 \text{ m} - 0.025 \text{ m} = 0.057 \text{ m}$. You are free to choose the step dy, but keep in mind that its value affects the smoothness of the profile: a too large step would make the profile too fragmented; a too small step makes indeed the profile smoother, but would require a longer computation time. For this exercise we suggest to set $dy = 0.01$ m for the first sections. The step of the last section will be slightly different to match the total Δy. In this case, we have five sections with $dy_{1-5} = 0.01$ m, which sum up to $\Delta y_{1-5} = 0.05$ m. Hence, the last step should be $dy_6 = \Delta y - \Delta y_{1-5} = 0.057 \text{ m} - 0.05 \text{ m} = 0.007 \text{ m}$. Given these steps, we can compute the different water depths as $y_i = y_{i-1} + dy_{i-1}$ with $i = 2, ..., 7$. The various points of the profile are found as $y_i = y_{i-1} + dy_{i-1}$.

Since the profile goes up, for the midpoint method we compute the midpoint water level as $y_{med,i} = y_i + \frac{dy_i}{2}$. As we will see in the following part of the exercise, if the profile is supposed to go down, the midpoint would be computed as $y_{med,i} = y_i - \frac{dy_i}{2}$.

For completing the profile we compute in order:

1. channel section $A_i = b \cdot y_{med,i}$
2. wetted perimeter $P_i = b + 2 \cdot y_{med,i}$
3. hydraulic radius $R_i = \frac{A_i}{P_i}$
4. flow velocity $v_i = \frac{Q}{A_i}$
5. slope of the profile $S_{f,i} = \left(\frac{n \cdot v_i}{R_i{}^{2/3}}\right)^2$
6. Froude number $Fr_i = \frac{v_i}{\sqrt{g \cdot y_{med,i}}}$

Be careful that all these parameters are calculated using $y_{med,i}$ and not y_i. This is relevant for the application of the middle point method.

Once these values are computed, we calculate the longitudinal steps according to the approximation

$$dx_i \approx \left(\frac{1 - Fr_i^2}{S_0 - S_{f,i}}\right) \cdot dy_i$$

from which we derive the longitudinal coordinates as $x_i = x_{i-1} + dx_{i-1}$.

Conjugate profile

The conjugate depths are found with the formula

$$y_{conj,i} = \frac{y_i}{2}\left(-1 + \sqrt{1 + 8Fr_i^2}\right)$$

All values are found in Table 2.

Table 2: Profile resulting from the gate opening and its conjugate.

Section	y_i (m)	dy (m)	y_{med} (m)	A (m^2)	P (m)	R (m)	v (m/s)	S_f (-)	Fr (-)	$S_0 - S_f$ (-)	$1 - Fr^2$ (-)	dx (m)	x (m)	h_{conj} (m)
1	0.025	0.01	0.030	0.0045	0.210	0.021	2.444	0.0850	4.506	-0.0840	-19.303	2.30	0	0.147
2	0.035	0.01	0.040	0.0060	0.230	0.026	1.833	0.0368	2.927	-0.0358	-7.566	2.12	2.30	0.128
3	0.045	0.01	0.050	0.0075	0.250	0.030	1.467	0.0195	2.094	-0.0185	-3.386	1.83	4.41	0.113
4	0.055	0.01	0.060	0.0090	0.270	0.033	1.222	0.0118	1.593	-0.0108	-1.538	1.43	6.24	0.099
5	0.065	0.01	0.070	0.0105	0.290	0.036	1.048	0.0078	1.264	-0.0068	-0.598	0.89	7.67	0.088
6	0.075	0.007	0.079	0.0118	0.307	0.038	0.934	0.0057	1.065	-0.0047	-0.133	0.20	8.55	0.081
7	0.082	-	-	-	-	-	-	-	-	-	-	-	8.75	0

Profile resulting from the long-crested weir

The water depth on the long-crested weir is not given, but we know that a choking occurs there. In this situation, the water depth is critical. As the channel section does not change, the critical depth also does not change compared to the one previously found. Hence, $y_{weir} = y_c = 0.082$ m.

Since we assume that no losses occur when flowing over the long-crested weir, we know that the specific energy upstream of the long-crested weir (section 1) and along it (section 2) must be equal. Hence, $E_1 = E_2 + d_{weir}$, from which we derive that $y_1 + \frac{v_1^2}{2g} = y_2 + \frac{v_2^2}{2g} + d_{weir}$. The profile is computed starting from y_1 going upstream, until the depth at the gate location. To get y_1, we first rewrite $v = \frac{Q}{bh}$, from which we obtain $y_1 + \frac{Q^2}{2g(by_1)^2} = y_2 + \frac{Q^2}{2g(by_2)^2} + d_{weir}$. Next, we make y_1 explicit. We obtain a 3rd order equation that reads $y_1^3 - y_1^2\left(y_2 + \frac{Q^2}{2g(by_2)^2} + d_{weir}\right) + \frac{Q^2}{2gb^2} = 0$. Since this is a 3rd order equation, it has three solutions for y_1: $y_1\prime = -0.0377$ m, $y_1\prime\prime = 0.0517$ m, and $y_1\prime\prime\prime = 0.139$ m. The negative solution has no physical sense and we already neglect it. The two other solutions are positive and identify a torrential regime ($y_1\prime\prime$) and a fluvial regime ($y_1\prime\prime\prime$). As this depth is found downstream of the hydraulic jump, we expect to be in subcritical conditions. Therefore, we pick $y_1 = y_1\prime\prime\prime = 0.139$ m.

At this point we apply a similar procedure to the one previously used for computing the profile resulting from the gate opening. We set a constant step $dy = 0.005$ m to compute the various points of the profile. You can choose any value, but the remarks previously mentioned still hold. Since this time we compute the profile from the highest point to the lowest, we invert the sign. For this, we get $y_i = y_{i-1} - dy$, with $i = 2, ..., n$, and consequently $y_{med,i} = y_i - \frac{dy}{2}$. As before, we compute $A_i = b \cdot y_{med,i}$, $P_i = b + 2 \cdot y_{med,i}$, $R_i = \frac{A_i}{P_i}$, $v_i = \frac{Q}{A_i}$, $S_{f,i} = \left(\frac{n \cdot v_i}{R_i{}^{2/3}}\right)^2$, and $Fr_i = \frac{v_i}{\sqrt{g \cdot y_{med,i}}}$. Eventually, $dx_i \approx \left(\frac{1-Fr_i^2}{S_0 - S_{f,i}}\right) \cdot dy_i$. Remember that, as we are computing the profile from downstream to upstream, all dx_i should be negative. The longitudinal coordinates are found similarly as before $x_i = x_{i-1} + dx_{i-1}$. Remember that the first point $x_1 = x_{weir} = 15$ m.

All results are included in Table 3.

Table 3: Profile resulting from the long-crested weir.

Section	y_i **(m)**	dy **(m)**	y_{med} **(m)**	A **(m^2)**	P **(m)**	R **(m)**	v **(m/s)**	S_f **(-)**	Fr **(-)**	$S_0 - S_f$ **(-)**	$1 - Fr^2$ **(-)**	dx **(m)**	x **(m)**
1	0.139	0.002	0.138	0.0207	0.426	0.049	0.531	0.00135	0.457	-0.00035	0.791	-4.55	15.00
2	0.137	0.002	0.136	0.0204	0.422	0.048	0.539	0.00140	0.467	-0.00040	0.782	-3.94	10.45
3	0.135	0.002	0.134	0.0201	0.418	0.048	0.547	0.00145	0.477	-0.00045	0.772	-3.43	6.52
4	0.133	0.002	0.132	0.0198	0.414	0.048	0.556	0.00150	0.488	-0.00050	0.762	-3.02	3.08
5	0.131	0.002	0.130	0.0195	0.410	0.048	0.564	0.00156	0.500	-0.00056	0.750	-2.67	0.06
6	0.129	0.002	0.128	0.0192	0.406	0.047	0.573	0.00162	0.511	-0.00062	0.739	-2.37	-2.60

Jump location

To localize the hydraulic jump, we look for the longitudinal coordinates where the conjugate profile and the one resulting from the long-crested weir meet. We see that the intersection point is slightly smaller than 0.14 m. To get the exact water depth, we can assume that both profiles are straight lines between the two points closest to the intersection point.

For the conjugate profile, the two points are $y_0 = 0.147$ m and $y_1 = 0.128$ m, and the related coordinates are $x_0 = 0.0$ m and $x_1 = 2.299$ m. Hence, $\Delta y = y_1 - y_0 = 0.128$ m -0.147 m$= -0.019$ m; $\Delta x = x_1 - x_0 = 2.299$ m -0.0 m$= 2.299$ m.

For the long-crested weir profile, the two points are $y_0 = 0.133$ m and $y_1 = 0.135$ m, and the related coordinates are $x_0 = 3.082$ m and $x_1 = 6.516$ m. Hence, $\Delta y = y_1 - y_0 = 0.135$ m -0.133 m$= 0.002$ m; $\Delta x = x_1 - x_0 = 6.516$ m -3.082 m$= 3.434$ m.

We then compute the slope of the lines as $m = \frac{\Delta y}{\Delta x}$, and the intercept as $q = y_0 - x_0 \cdot m$. For the conjugate profile we obtain $m_{conj} = -0.0082$ and $q_{conj} = 0.1473$. For the long-crested weir profile we obtain $m_{weir} = 0.0006$ and $q_{weir} = 0.1312$.

The location of the jump is found at the intersection between the two lines, that is where $x_{jump} \cdot m_{conj} + q_{conj} = x_{jump} \cdot m_{weir} + q_{weir}$, from which we make x explicit and get $x_{jump} = \frac{q_{weir} - q_{conj}}{m_{conj} - m_{weir}} = \frac{0.1312 - 0.1473}{-0.0082 - 0.0006} = 1.8296$ m.

The related depth can be found by using one or the other equations of the lines. Using the equation of the conjugate profile, for instance, we get $y_{jump} = x_{jump} \cdot m_{conj} + q_{conj} = 1.8926$ m $\cdot(-0.0.082) + 0.1473$ m $= 0.1323$ m. The same value can be found using the equation resulting from the long-crested weir profile.

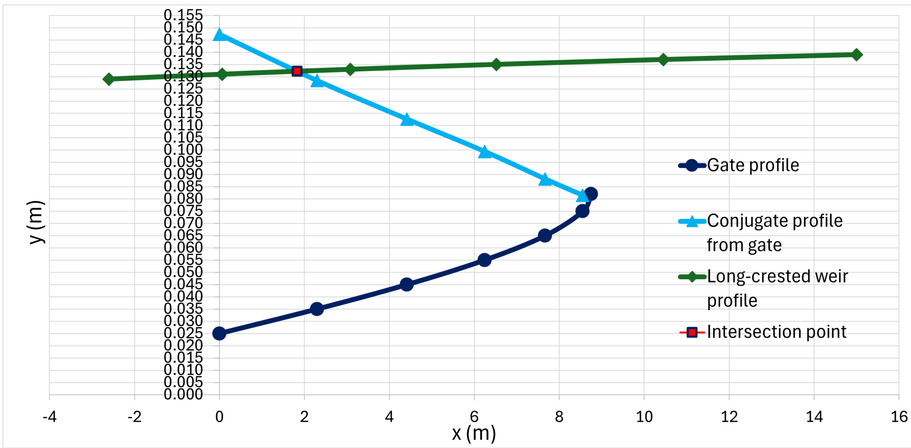

Figure 4: Profiles resulting from the gate opening, its conjugate, and from the long-crested weir. The intersection point between the conjugate and the long-crested weir profile results in the jump location.